Improved Euler Method Notes

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1 Improved Euler Method

The improved Euler method is defined by the equation

$$y_{i+1} = y_i + \frac{h}{2}(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))),$$

which can be conveniently calculated with the following equations:

$$k_{1i} = f(x_i, y_i),$$
$$k_{2i} = f(x_i + h, y_i + hk_{1i}),$$

$$y_{i+1} = y_i + \frac{h}{2}(k_{1i} + k_{2i}).$$

With the improved Euler method, we require two computations of f per step, instead of one (as in regular Euler). The regular Euler method provides us with a linear approximation to our solution (taking small steps along the tangent line at each point, approximating the curve as a set of connected lines). Improved Euler, however, provides us with a quadratic approximation. We take the average of the slopes of the endpoints of each subinterval. You could think of improved Euler as $y_{i+1} = y_i +$ trapezoid rule.

2 Sample Problem

Use the improved Euler method to find approximate values of the solution for the following problem:

$$y' = -2y + x^3e^{-2x},$$
$$y(0) = 1, \qquad \text{for } x = 0.1, 0.2$$

Solution.

We start with $x_0 = 0$ and, by the initial value condition, have $y_0 = 1$.

When x is equal to 0, y = 1.

We can now solve using the improved Euler equations.

$$k_{10} = f(x_0, y_0) = f(0, 1) = -2$$
$$k_{20} = f(x_1, y_0 + hk_{10}) = f(0.1, 1 + (0.1)(-2)) = f(0.1, 0.8) = -2(0.8) + (0.1)^3e^{-.2} = -1.5992...$$
$$y_1 = y_0 + \frac{h}{2}(k_{10} + k_{20}) = 1 + (0.05)(-2 - 1.5992...) = 0.82004...$$

When x is equal to 0.1, y ≈ .82004.

$$k_{11} = f(x_1, y_1) = f(0.1, .82004...) = -2(0.82004...) + (0.1)^3e^{-0.2} - 1.6393...$$
$$k_{21} = f(x_2, y_1 + hk_{11}) = f(0.2, 0.82004... + (0.1)(-1.6393...)) = f(0.2, 0.65611...) = -2(0.65611...) + (.2)^3e^{-0.4}$$
$$= -1.3069...$$
$$y_2 = y_1 + \frac{h}{2}(k_{11} + k_{21}) = 0.82004... + (0.05)(-1.6393... - 1.3069...) = 0.67273...$$

When x is equal to 0.2, y ≈ .67273.

And we are done!