Feb 21, 2024 Wksht 4.3

2^{nd}-order linear DE's (homogenous) & constant coefficients

Find the Solution

$$ay'' + by' + cy = 0$$

$$y'' + P(x)y' + H(x)y = G(x)$$

$$G(x) \to 0$$

$$ay'' + by' + cy = 0$$

$$ar^2 + br + c = 0$$

characteristic eqn.

Solve r. $\Rightarrow$ 2 real roots

Solution: $\boxed{C_1 e^{rt} + C_2 e^{rt}}$

$$ay'' + by' + cy = 0$$

$$x' = Ax = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}$$

$$= \frac{1}{a}(a\lambda^2 + b\lambda + c)$$

$$a\lambda^2 + b\lambda + c = 0$$

λ is an eigenvalue of A

$$-\lambda\gamma_1 + \gamma_2 = 0$$

$$v = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

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if λ is a root:

$$r = e^{\lambda t} v = \begin{pmatrix} e^{\lambda t} \\ \lambda e^{\lambda t} \end{pmatrix}$$

$y = e^{\lambda t}$ is a solution.

$$ay'' + by' + cy = 0$$

1) $ar^2 + br + c = 0$

solve for r

Solution is $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Ex.

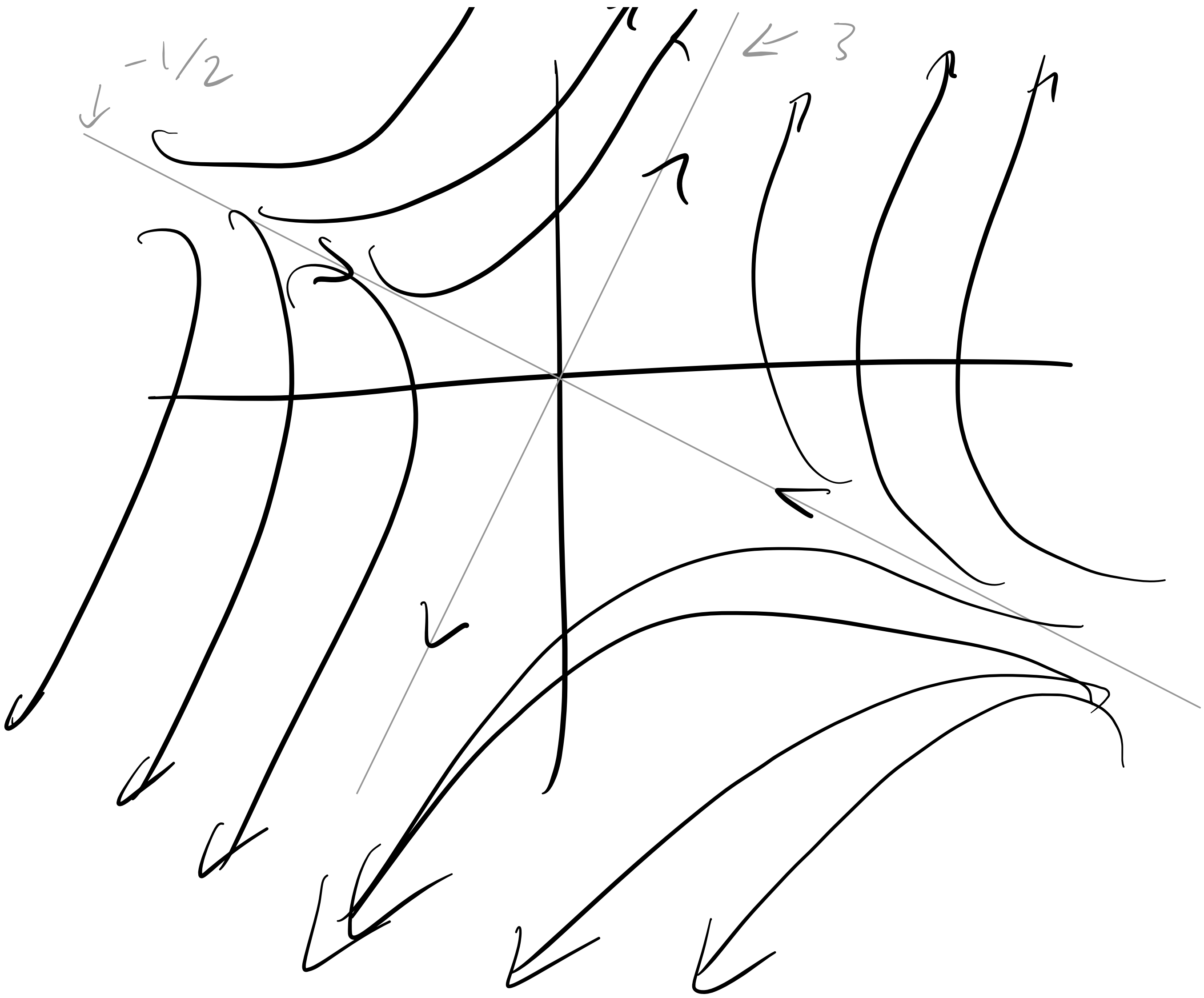
$$2y'' - 5y' - 3y = 0.$$

Solve

$$2r^2 - 5r - 3 = 0$$

$$r = 3, -\frac{1}{2}$$

Solution $=$ $y = C_1 e^{3t} + C_2 e^{-\frac{1}{2}t}$



-1/2
3

Ex $y'' - 10y' + 25y = 0$

$r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$

$r = 5, 5$

$y^3 \ldots$

$C_1e^{5t} + C_2e^{5t}t + C_3e^{5t}t^2$

Solution

$$\boxed{y = C_1 e^{5t} + C_2 e^{5t} t}$$

Ex. $4y'' + 4y' + 17y = 0$

$$4r^2 + 4r + 17 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(4)(17)}}{8}$$

$$r = \frac{-4 \pm \sqrt{-256}}{8}$$

$$r = \frac{-4 \pm 16i}{8}$$

$$r = \frac{-1}{2} \pm 2i$$

$$\frac{-1}{2} \pm 2i$$

G.S.

$$\boxed{y = e^{-\frac{1}{2}t}\left(C_1\cos(2t) + C_2\sin(2t)\right)}$$

2nd-order
homogenous
linear
ODE's
w/ constant coefficients

$ay'' + by' + cy = 0$ $\quad r$ roots of C.E.

Real: $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Repeated: $C_1 e^{r_1 t} + t C_2 e^{r_2 t}$

Complex: $e^{\alpha t}\left(C_1\cos(\beta t) + C_2\sin(\beta t)\right)$

$r = \alpha + \beta i$

Cauchy-Euler Equation:

$y(x)$

$$ax^2y'' + bxy' + cy = 0$$

$\frac{d^2y}{dx^2}$

$$a \neq 0, \quad x > 0.$$

Let $t = \ln(x)$.

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

$$\Downarrow$$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

$$ay'' + (b-a)y' + cy = 0$$

Solve. $ax^2y'' + bxy' + cy = 0$

Let $t = \ln(x)$

$$ay'' + (b-a)y' + cy = 0.$$

If $y(t)$ is a solution,

$y(\ln(x))$ is a solution

Ex. $3x^2 y'' + 4xy' + y = 0,$

$x > 0.$

Let $t = \ln(x)$

("eleven plus two"
"twelve plus one"
ARE ANAGRAMS!!!)

$$ay''(t) + (b-a)y'(t) + cy(t) = 0$$

$$3y''(t) + 1y'(t) + 1y(t) = 0$$

$$3\lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-4(3)}}{2(3)}$$

$$= \frac{-1 \pm \sqrt{-11}}{6}$$

$$= \frac{-1}{6} \pm \frac{\sqrt{11}\, i}{6}$$

$$y = e^{-\frac{1}{6}t}\left(c_1\cos\left(\frac{\sqrt{11}}{6}t\right) + c_2\sin\left(\frac{\sqrt{11}}{6}t\right)\right)$$

$t = \ln(x)$:

$$\boxed{y(x) = x^{-1/6}\left(c_1\cos\left(\frac{\sqrt{11}}{6}\ln(x)\right) + c_2\sin\left(\frac{\sqrt{11}}{6}\ln(x)\right)\right)}$$

Solved.

ay'' — — — —

$\frac{d^2y}{dx^2}$

ay'' — — —

$\frac{d^2y}{dt^2}$

$y^{(4)} + 2y'' + y = 0$

$r^4 + r^2 + 1 = 0$

this

Made up

$r = 2, 3, 4, 3$

$y = C_1 e^{2t} + C_2 e^{3t} + C_3 e^{4t} + C_4 e^{3t} t$

Let's say

roots are

$1 \pm 2i \quad , \quad 3 \pm 4i$

$C_1 e^{1t} \cos(2t) + C_2 e^{1t}(\sin(2t))$

$+ C_3 e^{3t} \cos(4t) + C_4 e^{3t}(\sin(4t)))$