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Laplace Transforms $\to$ Differential Eqn.

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

Ex. $y'' + 2y' + y = 5$, $y(0)=2$ $y'(0)=0$

$y = ?$

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{5\}$$

$$s^2 \ldots \quad s \ldots \quad Y(s) \ldots = \frac{5}{s}$$

Solve for $Y(s)$:

$$Y(s) = \frac{\text{-----}}{\text{-----}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t)$$

Ex.

$$y' - y = e^{3t}, \quad y(0) = 2$$

$$\overbrace{sY(s) - y(0)} - Y(s) = \frac{1}{s-3}$$

$$sY(s) - 2 - Y(s) = \frac{1}{s-3}$$

$$Y(s)(s-1) - 2 = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{(s-3)(s-1)} + \frac{2}{s-1}$$

$$\frac{1}{(s-3)(s-1)} = \frac{A}{s-1} + \frac{B}{s-3}$$

$$1 = (s-3)A + (s-1)B$$

$$1 = As - 3A + Bs - 1B$$

$$1 = s(A+B) - 3A - B$$

$A + B = 0$ $\quad$ $A = -B$

$-3A - B = 1$

$-3(-B) - B = 1$

$3B - B = 1$

$2B = 1$

$B = \frac{1}{2}$ $\quad$ $A = -\frac{1}{2}$

$$\frac{-1/2}{s-1} + \frac{1/2}{s-3}$$

$$= -\frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right)$$

$$Y(s) = -\frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right) + \frac{2}{s-1}$$

$$y(t) = -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} + 2e^{t}$$

$$\boxed{y(t) = \frac{3}{2}e^{t} + \frac{1}{2}e^{3t}}$$

Find the Laplace transform of

$$(2t+1)u_2(t).$$

$$f(t-c)\,u_c(t)$$

$$f(t-2) = 2t+1 \checkmark$$

$$f(t) = 2t+5$$

Alternatively

$$(2t+1)\,u_2(t)$$

$$= \left(2(t-2+2)+1\right)u_2(t)$$

$$(2(t-2)+4(t+1))\,u_2(t)$$

$$(2(t-2)+5)\,u_2(t) = f(t-2)\,u_2(t)$$

$$(2t+5)\,u_2(t) = f(t)\,u_2(t)$$

$$\boxed{\left(\frac{2}{s^2}+\frac{5}{s}\right)\frac{e^{-2s}}{s}}$$

L'

Ex.

$$y'' + 8y' + 2y = 0$$

$$y(0)=1$$

$y'(0)=0,$

Solve for $y(t) = ?$

$$\boxed{Y(s) = \frac{s+8}{s^2+8s+2}}$$

We'll stop there.

Ex.

inverse Laplace

$$Y(s) = \frac{s+9}{s^2+6s+5}$$

$$\boxed{y(t) = 2e^{-t} - e^{-5t}}$$