Math 2552 Final Review

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Problem 1
Solve the differential equation:

$$\frac{dy}{dt} = e^{5t+4y}$$

Solution:
First, we use the exponent product rule:

$$\frac{dy}{dt} = e^{5t}e^{4y}.$$

It is relatively clear that we can easily use the separation of variables method. First, we use the negative exponent property so that each of two variables appear on different sides of the equation:

$$e^{-4y}dy = e^{5t}dt.$$

We integrate both sides

$$\int e^{-4y}dy = \int e^{5t}dt,$$

and obtain

$$-\frac{1}{4}e^{-4y} = \frac{1}{5}e^{5t} + c_1.$$

We rearrange the terms to obtain

$$5e^{-4y} + 4e^{5t} = c_2.$$

Problem 2
Solve the differential equation:

$$\frac{dy}{dt} - 3y = t^2$$

Solution:
To solve the given differential equation, we can use the integrating factor method. The standard form of a first-order linear ODE is $\frac{dy}{dt} + P(t)y = Q(t)$, where $P(t)$ and $Q(t)$ are functions of t.

Comparing with the given equation, we have $P(t) = -3$ and $Q(t) = t^2$. The integrating factor, denoted by $\mu(t)$, is given by:

$$\mu(t) = e^{\int P(t)dt} = e^{\int(-3)dt} = e^{-3t}$$

Multiply both sides by the integrating factor:

$$e^{-3t}\frac{dy}{dt} - 3e^{-3t}y = t^2e^{-3t}$$

Now, notice that the left side is the result of the product rule, i.e., $\frac{d}{dt}(e^{-3t}y)$. Integrate both sides:

$$\int \frac{d}{dt}(e^{-3t}y)dt = \int t^2e^{-3t}dt.$$

The integral of the LHS is obvious. To integrate the RHS, use integration by parts[1] and obtain:

$$e^{-3t}y = -\frac{\left(9t^2+6t+2\right)\mathrm{e}^{-3t}}{27} + c_1.$$

[1] It may be good to review integration by parts! Do a few practice problems so that you are comfortable with it.

Solve for y by dividing both sides by e^{-3t}:

$$y = -\frac{(9t^2 + 6t + 2)}{27} + c_1 e^{3t}.$$

Problem 3
A population contains 10,000 bunnies, increasing at an annual growth rate of 12%. Every month, however, the population decreases by 50 bunnies.

Let $p(t)$ be the current number of bunnies at time t. Let r be the growth rate for the compounding period. Write the differential equation and initial condition for $p(t)$. Do not solve.

Solution:
First, we note that the population initially consisted of 10,000 bunnies. We can model this with the initial condition

$$p(0) = 10,000.$$

Now, we can set up the differential equation as

$$\frac{dp}{dt} = \text{bunnies accumulated} - \text{bunnies lost}.$$

The annual number of bunnies lost is $12(50) = 600$. In addition, the number of bunnies accumulated is equal to the growth rate for the compounding period times the current population, or $.12p$. Therefore, we have our solution:

$$\frac{dp}{dt} = .12p - 600, \qquad p(0) = 10,000.$$

Problem 4.1
Consider the following system of linear differential equations:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Find the general solution, plot the phase portrait, and classify the stability.

Solution:
To find the general solution, we start by finding the eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$.
First, we find the eigenvalues (λ).
Set up and solve the characteristic equation: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\det\begin{bmatrix} 3-\lambda & 4 \\ 1 & 3-\lambda \end{bmatrix} = 0$$
$$(3-\lambda)^2 - 4 = 0$$
$$\lambda^2 - 6\lambda + 5 = 0$$

We have $\lambda_1 = 1, \lambda_2 = 5$.

Next, we solve for the eigenvectors ($\mathbf{v}$): For $\lambda_1 = 1$, solve $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$:

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system leads to the equations $2x_1 + 4y_1 = 0$ and $x_1 + 2y_1 = 0$. We observe that $x_1 = -2y_1$. Therefore, we have

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -2y_1 \\ y_1 \end{bmatrix},$$

and choosing an arbitrary $y = 1$, we have:

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

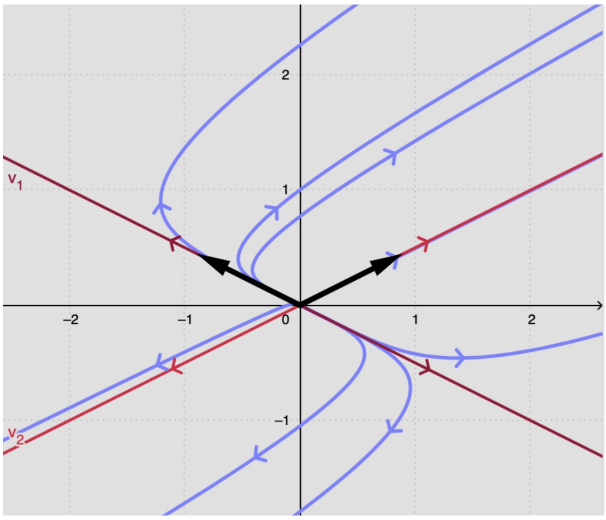
We repeat the same process for $\lambda = 5$ and obtain

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The general solution is given by:

$$\begin{aligned} \mathbf{X}(t) &= c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} \\ &= c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}. \end{aligned}$$

Next, we plot the phase portrait. We first draw our eigenvalues (denoted by red lines), or the lines $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$. Both of our eigenvalues are positive, so we know that the solution moves away from the origin (as $t \to \infty$, our exponential increases). $\mathbf{v_2}$ dominates, as it has the larger corresponding eigenvalue, so the solution is pulled in the direction of $\mathbf{v_2}$.



Two positive eigenvalues tells us that the solution is **unstable**.

Problem 4.2
Consider the following system of linear differential equations:

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Plot the phase portrait, and classify the stability. Do not solve for the general solution.

Solution:

We start by finding the eigenvalues (λ) of the coefficient matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}$.

Set up and solve the characteristic equation: $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$$\det \begin{bmatrix} 1-\lambda & 3 \\ -2 & -1-\lambda \end{bmatrix} = 0$$
$$(1-\lambda)(-1-\lambda) - 3(-2) = 0$$
$$\lambda^2 + 5 = 0$$

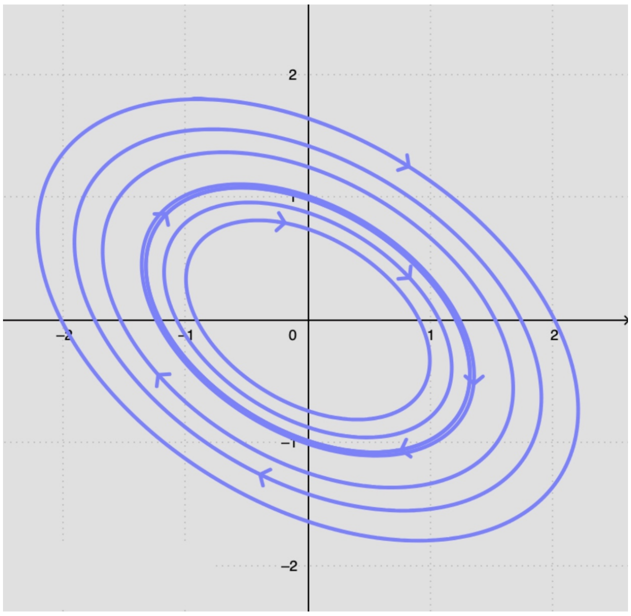
We have $\lambda = \pm\sqrt{-5} = \pm\sqrt{5}i$

Our eigenvalues are purely complex, which means that we will not have any real eigenvectors. As a result, our phase plane must be a circle or ellipse.

Now, we can plot our phase portrait. We need to identify the direction in which our ellipse is moving. We can sample points and plug them into our system to identify the direction in which the trajectory is travelling.

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies x' = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Hence, the ellipse is moving counterclockwise.



This system has two imaginary eigenvalues with no real counterpart, which means it can be considered a **stable ellipse**, in which trajectories never return to their steady-state and never converge, behaving as an undamped oscillator.

Problem 5.1
A LRC-series circuit has a capacitor of 0.50 farad and an inductor of 1 henry. If the initial charge on the capacitor is 3 coulomb, there is no initial current, and E(t) = 0, obtain the linear differential equation in $q(t)$ for the LRC series circuit. Do not solve.

Solution:
Recall the differential equation for an LRC-series circuit:

$$Lq''(t) + Rq'(t) + \frac{q(t)}{C} = E(t).$$

Substituting $L = 1$ henry, $R = 0$ ohms, $C = \frac{1}{2}$ farad, and $E(t) = 0$ yields the differential equation:

$$q''(t) + 2q(t) = 0.$$

The initial charge is 3 coulomb, so $q(0) = 3$. There is no initial current, so $q'(0) = 0$.
We may now write our solution:

$$q''(t) + 2q(t) = 0, \quad q(0) = 3, \quad q'(0) = 0.$$

Problem 5.2
A LRC-series circuit has a capacitor of 10^{-4} farad, a resistor of 20 ohms, and an inductor of .5 henry. If the initial charge on the capacitor is 1 coulomb and there is an initial current of 2 amperes, obtain the linear differential equation in $q(t)$ for the LRC series circuit. Do not solve.

Solution:
Recall the differential equation for an LRC-series circuit:

$$Lq''(t) + Rq'(t) + \frac{q(t)}{C} = E(t).$$

Substituting $L = .5$ henry, $R = 20$ ohms, $C = \frac{1}{10000}$ farad, and $E(t) = 0$ yields the differential equation:

$$0.5q''(t) + 20q'(t) + 10000q(t) = 0 \to q''(t) + 40q'(t) + 20000q(t) = 0$$

The initial charge is 1 coulomb, so $q(0) = 1$. The initial current is 2 amperes, and so $q'(0) = 2$.
We may now write our solution:

$$q''(t) + 40q'(t) + 20000q(t) = 0, \quad q(0) = 1, \quad q'(0) = 2.$$

Problem 6
Solve the differential equation:

$$3x^2y'' + 4xy' + y = 0, \quad x > 0$$

Solution:
Immediately, we can recognize that this is a Cauchy-Euler equation. We will use the substitution

$$t = \ln(x),$$

and we can transform the equation into

$$aY''(t) + (b - a)Y'(t) + cY(t) = 0.$$

Hence, we have

$$3Y''(t) + 1Y'(t) + Y(t) = 0.$$

Next, we solve the corresponding auxiliary equation:

$$3\lambda^2 + 1\lambda + 1 = 0.$$

We obtain

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4(3)(1)}}{2(3)} = \frac{-1 \pm \sqrt{-11}}{6} = -\frac{1}{6} \pm i\frac{\sqrt{11}}{6}.$$

Now, making use of Euler's formula, we write our solution as a function of t:

$$Y_c(t) = e^{-t/6}(c_1 \cos \frac{\sqrt{11}}{6}t + c_2 \sin \frac{\sqrt{11}}{6}t).$$

We substitute back $t = \ln(x)$ and obtain our final solution as a function of x:

$$y_c(x) = x^{-\frac{1}{6}}(c_1 \cos \frac{\sqrt{11}}{6} \ln x + c_2 \sin \frac{\sqrt{11}}{6} \ln x)$$

Problem 7
Find the general solution of the differential equation:

$$y'' - 4y' - 12y = te^{4t}$$

Use the method of underdetermined coefficients.

Solution:
The general solution is the sum of the complementary solution and the particular solution. First, we will find the complementary solution, y_c. We begin by solving

$$y'' - 4y' - 12y = 0.$$

The characteristic equation may be solved:

$$r^2 - 4r - 12r = (r-6)(r+2) = 0 \to r_1 = 6, r_2 = -2.$$

Since we have real roots, our complementary solution is

$$y_c = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{6t} + c_2 e^{-2t}.$$

Now, we need to find our particular solution. For a given differential equation of the form

$$y'' + p(t)y' + q(t)y = g(t),$$

we can use $g(t)$ to make an educated guess about the form of our particular solution. In this problem, $g(t) = te^{4t}$.

A good guess for the t term is $At + B$, and a good guess for the e^{4t} term is Ce^{4t}. We combine these by multiplying:

$$y_p = Ce^{4t}(At + B) = e^{4t}(CAt + CB) = e^{4t}(At + B),$$

where we can collapse constants.

With y_p, we can derive y_p' and y_p'':

$$\begin{aligned} y_p &= e^{4t}(At + B) \\ y_p' &= e^{4t}(4At + 4B + A) \\ y_p'' &= e^{4t}(16At + 16B + 8A). \end{aligned}$$

We can substitute these into our original problem:

$$\begin{aligned} e^{4t}(16At + 16B + 8A) - 4\left(e^{4t}(4At + 4B + A)\right) - 12\left(e^{4t}(At + B)\right) &= te^{4t} \\ (16A - 16A - 12A)\, te^{4t} + (16B + 8A - 16B - 4A - 12B)\, e^{4t} &= te^{4t} \\ -12Ate^{4t} + (4A - 12B)\, e^{4t} &= te^{4t} \end{aligned}$$

We can now solve by setting the coefficients of the terms on both sides equal (note that there is one te^{4t} term on the right and zero e^{4t} terms on the right):

$$\begin{aligned} te^{4t} &: -12A = 1 \quad \Rightarrow A = -\frac{1}{12} \\ e^{4t} &: 4A - 12B = 0 \quad \Rightarrow B = -\frac{1}{36} \end{aligned}$$

With our constants A and B, we may now write our particular solution:

$$y_p(t) = \mathbf{e}^{4t}\left(-\frac{t}{12} - \frac{1}{36}\right) = -\frac{1}{36}(3t+1)\,\mathbf{e}^{4t}$$

Recall the complementary solution:

$$y_c = c_1 e^{6t} + c_2 e^{-2t}$$

Our general solution is therefore

$$y = c_1 e^{6t} + c_2 e^{-2t} - \frac{1}{36}(3t+1)\,\mathbf{e}^{4t}.$$

Problem 8
Find the general solution of the differential equation:

$$y'' + 16y = 2\sin(4t)$$

Note that the complementary solution is

$$y_c = c_1\cos(4t) + c_2\sin(4t)$$

Solve using the method of underdetermined coefficients.

Solution:
The general solution is the sum of the complementary solution and the particular solution. And in this problem, we are already given the complementary solution!

$$y_c = c_1\cos(4t) + c_2\sin(4t)$$

Next, we will find our particular solution. Like the last problem, for a given differential equation of the form

$$y'' + p(t)y' + q(t)y = g(t),$$

we can use $g(t)$ to make an educated guess about the form of our particular solution. In this problem, $g(t) = 2\sin(4t)$.

A good guess for the particular solution is $A\cos(4t) + B\sin(4t)$. However, note that the complementary solution already contains solutions of this form. Hence, we will include a t term in our particular solution:

$$y_p = At\cos(4t) + Bt\sin(4t)$$

With y_p, we can derive y_p' and y_p'':

$$y_p = At\cos(4t) + Bt\sin(4t)$$

$$y_p' = A\cos(4t) - 4At\sin(4t) + B\sin(4t) + 4Bt\cos(4t)$$
$$y_p'' = -16A\sin(4t) - 16At\cos(4t) + 16B\cos(4t) - 16Bt\sin(4t)$$

We can substitute these into our original problem:

$$(-16A\sin(4t) - 16At\cos(4t) + 16B\cos(4t) - 16Bt\sin(4t)) + 16(At\cos(4t) + Bt\sin(4t)) = 2\sin(4t)$$
$$-16A\sin(4t) + 16B\cos(4t) = 2\sin 4(t)$$

We can now solve by setting the coefficients of the terms on both sides equal (note that there are two $\sin 4(t)$ terms on the right and zero $\cos(4t)$ terms on the right):

$$\sin 4(t) : -16A = 2 \quad \Rightarrow A = -\frac{1}{8}$$
$$\cos 4(t) : 16B = 0 \quad \Rightarrow B = 0$$

With our constants A and B, we may now write our particular solution:

$$y_p = -\frac{1}{8}t\cos(4t)$$

Recall the complementary solution:

$$y_c = c_1\cos(4t) + c_2\sin(4t)$$

$$y = c_1\cos(4t) + c_2\sin(4t) - \frac{1}{8}t\cos(4t)$$

Problem 9.1
Find the particular solution of the differential equation:

$$y'' + 16y = 2\tan(4t), \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Note that the complementary solution is $c_1\cos(4t) + c_2\sin(4t)$.

Solve using variation of parameters.

Solution:
Our fundamental matrix can be computed as the Wronskian of our solutions (given by the complementary equation):

$$\phi = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{bmatrix}$$

Our particular solution is

$$y_p = u_1y_1 + u_2y_2 = u_1\cos(4t) + u_2\sin(4t),$$

where

$$u_1 = \int -\frac{y_2 g(t)}{|\phi|}, \quad u_2 = \int \frac{y_1 g(t)}{|\phi|}$$

and $g(t)$ is equal to the right hand side of our equation, or $2\tan(4t)$.
Solving, we have

$$u_1 = -\int \frac{2\sin(4t)\tan(4t)}{4\cos^2 + 4\sin^2} dt = \int \frac{2(\cos(4t) - \sec(4t))}{4} dt = \frac{1}{2} \int (\cos(4t) - \sec(4t)) dt$$
$$= \frac{1}{2} \frac{(\sin(4t) - \ln|\sec(4t) + \tan(4t)|)}{4} = \frac{(\sin(4t) - \ln|\sec(4t) + \tan(4t)|)}{8}$$

Note that from the first step to the second, we make use of the trigonometric transformation:

$$-2\sin(4t)\tan(4t) = 2(\cos(4t) - \sec(4t))$$

This can be derived with the following steps:

$$\begin{aligned}
&-2\sin(4t)\tan(4t) \\
&= -2\sin(4t)\frac{\sin(4t)}{\cos(4t)} \\
&= -2\sin^2(4t)\frac{1}{\cos(4t)} \\
&= -2(1 - \cos^2(4t))\frac{1}{\cos(4t)} \\
&= -2(\sec(4t) - \cos(4t)) \\
&= 2(\cos(4t) - \sec(4t)).
\end{aligned}$$

$$u_2 = \int \frac{2\cos(4t)\tan(4t)}{4\cos^2 + 4\sin^2} dt = \int \frac{2\sin(4t)}{4} dt = \frac{1}{2} \int (\sin(4t)) dt$$
$$= -\frac{1}{2} \frac{\cos(4t)}{4} = -\frac{\cos(4t)}{8}$$

Thus,

$$y_p = u_1 \cos(4t) + u_2 \sin(4t) = \frac{(\sin(4t) - \ln|\sec(4t) + \tan(4t)|)}{8} \cos(4t) - \frac{\cos(4t)}{8} \sin(4t).$$

Problem 9.2
Find the particular solution of the differential equation:

$$y'' + 16y = 2\sec(4t), \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Note that the complementary solution is $c_1 \cos(4t) + c_2 \sin(4t)$.

Solve using variation of parameters.

Solution:
Our fundamental matrix can be computed as the Wronskian of our solutions (given by the complementary equation):

$$\phi = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{bmatrix}$$

Our particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = u_1 \cos(4t) + u_2 \sin(4t),$$

where

$$u_1 = \int -\frac{y_2 g(t)}{|\phi|}, \quad u_2 = \int \frac{y_1 g(t)}{|\phi|}$$

and $g(t)$ is equal to the right hand side of our equation, or $2\sec(4t)$.
Solving, we have

$$u_1 = -\int \frac{2\sin(4t)\sec(4t)}{4\cos^2 + 4\sin^2} dt = -\int \frac{2\tan(4t)}{4} dt = -\frac{1}{2}\int \tan(4t) dt$$
$$= -\frac{1}{2}(-\frac{1}{4}\ln|\cos(4t)|) = \frac{\ln|\cos(4t)|}{8}$$
$$u_2 = \int \frac{2\cos(4t)\sec(4t)}{4\cos^2 + 4\sin^2} dt = \int \frac{2}{4} dt = \frac{t}{2}$$

Thus,

$$y_p = u_1\cos(4t) + u_2\sin(4t) = \frac{\ln|\cos(4t)|}{8}\cos(4t) - \frac{t}{2}\sin(4t).$$

Problem 10
Find the Laplace transform of the solution of the differential equation:

$$y'' - 8y' = 2e^{2t} - 4e^{-t} + \cos(5t), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution:

First, we apply the Laplace operator $\mathcal{L}$ to each term:

$$\mathcal{L}\{y''\} - 8\mathcal{L}\{y'\} = 2\mathcal{L}\{e^{2t}\} - 4\mathcal{L}\{e^{-t}\} + \mathcal{L}\{\cos(5t)\}$$

Then, we apply the Laplace transform for each of these terms, obtaining

$$s^2Y(s) - sy(0) - y'(0) - 8sY(s) + 8y(0) = \frac{2}{s-2} - \frac{4}{s+1} + \frac{s}{s^2+25}.$$

Next, we use the initial conditions $y(0) = 1$ and $y'(0) = -1$:

$$s^2Y(s) - s + 1 - 8sY(s) + 8 = \frac{2}{s-2} - \frac{4}{s+1} + \frac{s}{s^2+25}$$
$$\to s^2Y(s) - 8sY(s) = \frac{2}{s-2} - \frac{4}{s+1} + \frac{s}{s^2+25} + s - 9$$
$$\to Y(s)(s^2 - 8s) = \frac{2}{s-2} - \frac{4}{s+1} + \frac{s}{s^2+25} + s - 9$$
$$\to Y(s) = \frac{2}{s(s-2)(s-8)} - \frac{4}{s(s+1)(s-8)} + \frac{s}{s(s^2+25)(s-8)} + \frac{s-9}{s(s-8)}$$

This problem asks for the Laplace transform of the solution, not the solution itself as a function of t, so we are done.

Problem 11
Find the Laplace transform of $(2t+1)u_2(t)$.

Solution:

We would like to solve $\mathcal{L}\{(2t+1)u_2(t)\}$.
Recall that $u_c(t) = u(t-c)$. We can rewrite the problem as

$$\mathcal{L}\{(2t+1)u(t-2)\}$$

We have that u is a function of $t-2$, so we would like the first term (2t+1) to be of a similar form. We can write our function

$$\begin{aligned} f(t) &= (2t+1)u(t-2) \\ &= (2(t-2+2)+1)u_2(t) \\ &= (2(t-2)+4+1)u_2(t) \\ &= (2(t-2)+5)u_2(t) \end{aligned}$$

Recall the rule

$$\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}.$$

Using this rule, we obtain

$$\begin{aligned} F(s) &= e^{-2s}\mathcal{L}\{2t+5\} \\ &= e^{-2s}(2\mathcal{L}\{t\} + \mathcal{L}\{5\}) \\ &= e^{-2s}(2\frac{1!}{s^{1+1}} + \frac{5}{s}) \\ &= e^{-2s}(\frac{2}{s^2} + \frac{5}{s}). \end{aligned}$$

Problem 12
Find the inverse Laplace transform of the following expression:

$$\frac{s+9}{s^2+6s+5}$$

Solution:

First, we can factor the denominator:

$$\frac{s+9}{s^2+6s+5} = \frac{s+9}{(s+1)(s+5)}$$

Now, we can perform partial fraction decomposition:

$$\frac{s+9}{(s+1)(s+5)} = \frac{A}{s+1} + \frac{B}{s+5}$$

We solve

$$\begin{aligned} s+9 &= A(s+5) + B(s+1) \\ s+9 &= As + 5A + Bs + 1B \\ s+9 &= s(A+B) + (5A+B) \end{aligned}$$

We solve for like terms, so we have

$$A + B = 1, \quad 5A + B = 9$$

Solving the system of equations gives us

$$A = 2, B = -1$$

Now, we plug in our values for A and B:

$$\frac{s+9}{(s+1)(s+5)} = \frac{2}{s+1} - \frac{1}{s+5}$$
$$= 2\frac{1}{s+1} - 1\frac{1}{s+5}$$

So,

$$y(t) = \mathcal{L}^{-1}\{2\frac{1}{s+1} - 1\frac{1}{s+5}\}$$
$$= 2\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{1}{s+5}\}$$
$$= 2e^{-t} - e^{-5t}.$$

Problem 13A
Find the critical points of the following system:

$$x' = x(3 - x - 2y) \tag{1}$$
$$y' = y(4 - 2y - 4x). \tag{2}$$

Solution:

Note that there are two terms in (2): "y" and "$4 - 2y - 4x$". We can set both terms equal to 0 to find our solutions.

First, let us consider the first term in (2), for which we have $y = 0$. Let us substitute $y = 0$ into (1), and we solve:

$$x(3 - x - 2(0)) = 0$$
$$x(3 - x) = 0$$
$$x = 0, x = 3.$$

We have $x = 0$, $x = 3$, and $y = 0$. We can now construct combinations of these solutions to identify two fixed points: $(0, 0)$ and $(3, 0)$.

Second, we will consider the other term in (2): $4 - 2y - 4x$. Here, a solution for y is not a single real value but is instead a function of x. We may write

$$4 - 2y - 4x = 0$$
$$4 - 4x = 2y$$
$$2 - 2x = y.$$

We may now substitute y into (1) to solve for x, and we have

$$x(3 - x - 2y) = 0$$

$$x(3 - x - 2(2 - 2x)) = 0$$
$$x(3 - x - 4 + 4x) = 0$$
$$x(-1 + 3x) = 0$$
$$x = 0, x = \frac{1}{3}.$$

We may substitute each solution of x into $y = 2 - 2x$ to find the corresponding y-values:

$$y = 2,$$

and

$$y = \frac{4}{3}.$$

Therefore, we have the following two critical points: (0,2) and $(\frac{1}{3}, \frac{4}{3})$.

In total, we have identified four critical points:

$$(0, 0)$$
$$(3, 0)$$
$$(0, 2)$$
$$(\frac{1}{3}, \frac{4}{3}),$$

and we are done.

Problem 13B[a]

Consider the system:

$$x' = x(3 - x^2 - 2y)$$

$$y' = y(4 - 8y - 4x)$$

Write the approximating linear system near (0, 2).

[a] We're considering a different system in 13B and 13C than we did for 13A. This is not usually the case, but it doesn't change the computations.

Solution:

First, we can rewrite x' and y':

$$x' = 3x - x^3 - 2xy$$
$$y' = 4y - 8y^2 - 4xy$$

Then, we compute the Jacobian matrix J of this system:

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 3 - 3x^2 - 2y & -2x \\ -4y & 4 - 16y - 4x \end{pmatrix}.$$

Now, we evaluate at (0,2):

$$\begin{pmatrix} 3 - 3(0)^2 - 2(2) & -2(0) \\ -4(2) & 4 - 16(2) - 4(0) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -8 & -28 \end{pmatrix}.$$

We may now write our solution:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -8 & -28 \end{pmatrix} \begin{pmatrix} x \\ y-2 \end{pmatrix}.$$

Problem 13C
Consider the system:

$$x' = x(3 - x^2 - 2y)$$

$$y' = y(4 - 8y - 4x)$$

Determine whether (0,2) is stable, asymptotically stable, or unstable.

Solution:

Recall that the solution from the previous question is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -8 & -28 \end{pmatrix} \begin{pmatrix} x \\ y-2 \end{pmatrix}.$$

We may now solve for the eigenvalues of our matrix. Remember that to solve for the eigenvalues for a matrix A, we first compute A - λI and then solve for the roots of a polynomial $ad - bc$.

For a 2 x 2 matrix, when either b or c is 0, our eigenvalues are equal to the diagonal elements (this is because $bc = 0$ so $ad - bc = ad - 0 = ad$. The roots of a and d can be found independently, and are equal to the diagonal). Note that this may not always be the case. If neither b nor c are zero, you must solve for the eigenvalues by solving for the characteristic equation $\det(A - \lambda I)$.

Therefore, our eigenvalues are -1 and -28. Since our eigenvalues are real, different, and negative, the system is **asymptotically stable**. If our eigenvalues were real, different, and positive, the system would be unstable. If both were real (but one is positive and one is negative), the system would be an unstable saddle point.

Problem 14
Consider the differential equation

$$y' = 2 + 2t - y, y(0) = 1$$

Find the approximate values at $t = 0.1$ using the Euler method with step size $h = 0.05$.

Solution:
We start with $x_0 = 0$ and, by the initial value condition, have $y_0 = 1$.

We start with $t_0 = 0, y_0 = 1$.

Recall the general form of first-order Euler's method:

$$y_{n+1} = y_n + h(2 + 2t_n - y_n)$$

We have

$$y_1 = y_0 + h(2 + 2t_0 - y_0) = 1 + 0.05(2 + 2(0) - 1) = 1 + 0.05(1) = 1.05$$
$$y_2 = y_1 + h(2 + 2t_1 - y_1) = 1.05 + 0.05(2 + 2(0.05) - 1.05) = 1.05 + 0.05(1.05)$$

When x is equal to 0.1, y $\approx$ 1.05 + 0.05(1.05) = 1.1025.

Problem 15
Consider the differential equation

$$y' = t + y, y(0) = 1$$

Find the approximate values at $t = 0.1$ using the improved Euler method with step size $h = 0.1$.

Solution:

We start with $t_0 = 0$ and, by the initial value condition, have $y_0 = 1$.

Recall the general form for the improved Euler method:

$$y_{i+1} = y_i + \frac{h}{2}(f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))),$$

which can be conveniently calculated with the following equations:

$$k_{1i} = f(t_i, y_i),$$
$$k_{2i} = f(t_i + h, y_i + hk_{1i}).$$

Now, we calculate $k_{10}, k_{20},$ and y_1:

$$k_{10} = f(t_0, y_0) = f(0, 1) = 1$$
$$k_{20} = f(t_1, y_0 + hk_{10}) = f(0.1, 1 + (0.1)(1)) = f(0.1, 1.1) = 1.2$$
$$y_1 = y_0 + \frac{h}{2}(k_{10} + k_{20}) = 1 + (0.05)(2.2) = 1 + 0.11 = 1.11$$

When x is equal to 0.1, y $\approx$ 1.11.

Problem 16
Find the inverse Laplace transform of the following expression using convolution:

$$\frac{1}{(s+5)^6(s^2+9)}$$

Solution:

Consider that

$$\frac{1}{(s+5)^6(s^2+9)} = \frac{1}{(s+5)^6}\frac{1}{(s^2+9)} = F(s)G(s),$$

where

$$F(s) = \frac{1}{(s+5)^6} \text{ and } G(s) = \frac{1}{(s^2+9)}.$$

Note that we have the following two inverse laplace transform rules:

$$H(s) = \frac{n!}{(s-a)^{n+1}} \to h(t) = t^n e^{at} \tag{1}$$

$$H(s) = \frac{a}{s^2+a^2} \to h(t) = \sin(at) \tag{2}$$

You should know how to obtain inverse laplace transforms from a combination of transforms on your formula sheet. For example, (1) is not on your formula sheet, but it is a combination of two transforms on your formula sheet.

We want to put $F(s)$ and $G(s)$ into an appropriate form, so that we use rules (1) and (2).
Note that if we multiply $F(s)$ by $\frac{5!}{5!}$, we can write:

$$F(s) = \frac{1}{5!}\frac{5!}{(s+5)^6}.$$

We may also multiply $G(s)$ by $\frac{3}{3}$, and we have

$$G(s) = \frac{1}{3}\frac{3}{(s^2+9)}.$$

We may now use rules (1) and (2), so that we have

$$f(t) = \frac{1}{5!}t^5 e^{-5t}$$

$$g(t) = \frac{1}{3}\sin(3t).$$

Recall the convolution equation:

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{3}$$

We may now substitute $f(t)$ and $g(t)$ into equation (3). First, we move our coefficients outside of the integral (by the property of linearity).

$$\frac{1}{5!}\frac{1}{3} = \frac{1}{360},$$

and we have

$$\frac{1}{360}\int_0^t (t-\tau)^5 e^{-5(t-\tau)}\sin(3\tau)d\tau.$$

.

Problem 17
A tank initially contains 200 gallons of fluid in which 20 pounds of salt are dissolved. Brine containing 3 pounds of salt per gallon is pumped into the tank at a rate of 5 gallons per minute. The well-mixed solution is pumped out at a rate of 5 gallons per minute. Let $Q(t)$ be the amount of salt in the tank at time t. Write the differential equation for $Q(t)$ and its initial condition. Do not solve.

Solution:
Recall

$$\frac{dQ}{dt} = c_i(t)r_i(t) - \frac{Q(t}{V(t)}r_o(t),$$

or

$$\frac{dQ}{dt} = \text{rate of salt entering } - \text{rate of salt leaving}$$

The rate of salt entering is 3(5) (3 pounds of salt pumping in at 5 gallons per minute). The rate going out is equal to $\frac{5}{200}Q(t)$, as we substitute $V = 200$ and $r_o = 5$. Initially (at time 0), there are 20 pounds of salt.

We can write our solution:

$$\frac{dQ}{dt} = 15 - \frac{Q(t)}{40}, \quad Q(0) = 20$$

Random things to know

Be sure to remember your general integration rules! Some of these may be helpful:

$\int \frac{1}{x} = \ln|x| + C$

$\int \frac{1}{1+x} = \ln|x+1| + C$

$\int \tan(x) = -\ln|\cos(x)| + C$

$\int \sec(x) = \ln|\sec(x) + \tan(x)| + C$

Also, remember how to do integration by parts!

For example, if we needed to find the integral of:

$$\int e^x \sin(x) dx$$

We use the formula

$$\int fg' = fg - \int f'g,$$

where $f = \sin(x)$, $f' = \cos x$, $g = e^x$ and $g' = e^x$.

We have

$$e^x \sin(x) - \int e^x \cos(x) dx$$

Now, we have to do it again, for the second term! We have $f = \cos(x)$, $f' = -\sin x$, $g = e^x$ and $g' = e^x$.

We can write the whole solution as

$$e^x \sin(x) - (e^x \cos(x) - \int -e^x \sin(x) dx),$$

or

$$e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x) dx.$$

Our original problem, $\int e^x \sin(x) dx$, can be denoted by A.

We have

$$A = e^x \sin(x) - (e^x \cos(x) - A),$$

or

$$2A = e^x \sin(x) - (e^x \cos(x)),$$

which reduces to

$$A = \frac{e^x \sin(x) - e^x \cos(x)}{2}.$$