Problems:

1) Find the general solution: (separable variables)

$$\frac{dy}{dx} = \frac{(e^{y}+2)^{2}e^{-y}}{(e^{x}+1)^{4}e^{-x}}$$

2) Find the general solution: (integrating factor)

$$\frac{dy}{dx} = 4y + 8x + 2 \,;\ y(0)=2 \qquad I.F. = e^{\int P(x)\,dx}$$

3) Determine the interval of existence:

$$ty'' + 3y = t \,; \qquad y(1)=1, \quad y'(1)=2$$

4) Find and classify all equilibrium points.

$$y' = y^{2}(1-y)$$

5) Write the following 2^{nd}-order differential equations as a system of first-order, linear differential equations.

$$2y'' - 5y' + y = 0 \qquad y(3)=6, \quad y'(3)=-1$$

6) Find the critical points and classify the stability.

$$y' = y^3 + 3y^2 + 2y.$$

7)

(Hint):

Two Real Roots: λ_1, λ_2

General Solution: $C_1 e^{\lambda_1 t}\eta_1 + C_2 e^{\lambda_2 t}\eta_2$

Two Complex Roots: $a \pm ib$

General Solution: $C_1 e^{at}\cos(bt) + C_2 e^{at}\sin(bt)$

Two Identical Roots (repeated):

General Solution: $C_1 e^{\lambda_1 t}\eta_1 + C_2 e^{\lambda_1 t}(t\eta_1 + \eta_2)$

where η_2 is the generalized eigenvector.

Real:

Two positive real λ: source (unstable)

Two negative real λ: Sink (stable)

One positive λ, one negative λ: saddle (unstable).

Complex $(a + ib)$:

If $a < 0 \Rightarrow$ spiral sink (stable)

$a > 0 \Rightarrow$ spiral source (unstable)

$a = 0 \Rightarrow$ Centers (neutral)

Repeated:

$\lambda > 0 \Rightarrow$ unstable degenerate node

$\lambda < 0 \Rightarrow$ stable degenerate node

a) Solve the general solution and IVP:

$$x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} x, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

(real eigenvalues)

b) Solve the general solution and IVP:

$$x' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

(complex eigenvalues)

c) Solve the general solution and IVP:

$$x' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} x, \quad \vec{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

(repeated eigenvalues)

8) Application / Modeling Questions

- Newton's Law of Cooling
- Population growth / decline
- Investment problem
- Single or multiple tank

There will be one modeling problem drawn from one of these.

$$\dot{x} = A\boxed{x} \implies \text{Differential eqn.}$$

$$\dot{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x$$

Solutions:

1) Find the general solution:

$$\frac{dy}{dx} = \frac{(e^y+2)^2 e^{-y}}{(e^x+1)^4 e^{-x}}$$

$$dy\,(e^x+1)^4 e^{-x} = (e^y+2)^2 e^{-y}\,dx$$

$$dy\,\frac{e^y}{(e^y+2)^2} = dx\,\frac{e^x}{(e^x+1)^4}$$

$\int \qquad\qquad \int$

$$\int dy \frac{e^y}{(e^y+2)^2} = \int dx \frac{e^x}{(e^x+1)^4}$$

Substitution

$$\left.\begin{aligned} u &= e^y + 2 \\ \frac{du}{dy} &= e^y \\ dy &= e^{-y}\,du \end{aligned}\right\}$$

original $\int dy \frac{e^y}{(e^y+2)^2}$

$$= \int e^{-y} \frac{e^y}{u^2}\,du$$

$$= \int du \frac{1}{u^2}$$

$$= -\frac{1}{u}$$

$$= -\frac{1}{e^y+2} + C_1$$

LHS

$$\left.\begin{aligned} u &= e^x + 1 \\ \frac{du}{dx} &= e^x \end{aligned}\right\}$$

$$dx = e^{-x}\,du$$

$$\int dx \frac{e^x}{(e^x+1)^4}$$

$$= \int e^{-x} \frac{e^x}{u^4}\,du$$

$$= \int du \frac{1}{u^4}$$

$$= -\frac{1}{3u^3}$$

$$= -\frac{1}{3(e^x+1)^3} + C_2$$

RHS

$$\boxed{\frac{1}{e^y+2} = \frac{1}{3(e^x+1)^3} + C}$$ ✓

General Solution

2) Find the general solution, solve the IVP.

$$\frac{dy}{dx} = 4y + 8x + 2 \; ; \; y(0) = 2$$

$$\frac{dy}{dx} + P(y) = T$$

$$\frac{dy}{dx} - 4y = 8x + 2 \qquad (1)$$

I.F.

$$\mu(x) = e^{\int p(x)\,dx}$$

$$P(x) = -4$$

$$e^{\int -4\,dx}$$

$$\mu(x) = e^{-4x}$$

$$\underbrace{e^{-4x}\left(\frac{dy}{dx}\right) - 4e^{-4x}y}_{\left(e^{-4x}y\right)'} = 8xe^{-4x} + 2e^{-4x}$$

Take the integral of both sides.

$$\int \left(e^{-4x}y\right)' = \int 8xe^{-4x} + 2e^{-4x}\,dx$$

$$\downarrow$$

$$\boxed{e^{-4x}y} = 2\int (4x+1)\,e^{-4x}\,dx$$

$$\int fg' = fg - \int f'g$$

$$f = 4x+1 \qquad g' = e^{-4x}$$

$$f' = 4 \qquad g = -\frac{e^{-4x}}{4}$$

RHS:

$$2\left[\frac{-(4x+1)e^{-4x}}{4} - \frac{e^{-4x}}{4} + c\right]$$

$e^{-4x}(\ldots)$

$$= -\frac{(4x+1)e^{-4x}}{2} - \frac{e^{-4x}}{2} + C$$

$$= -e^{-4x}\left(2x + \frac{1}{2} + \frac{1}{2}\right) + C$$

$$e^{-4x}y = -(2x+1)e^{-4x} + C$$

General Solution

$$\boxed{y = -(2x+1) + Ce^{4x}}$$

IVP:

$$y(0) = 2$$

$$2 = -(2(0)+1) + Ce^{4(0)}$$

$$2 = -(1) + Ce^{0}$$

$$2 = -1 + C$$

$$\underline{C = 3}$$

$$\boxed{y = -(2x+1) + 3e^{4x}}$$

IVP Solution

3) Determine the interval of existence:

$$ty'' + 3y = t \;; \quad \boxed{\begin{array}{l} y(1) = 1 \\ y'(1) = 2 \end{array}}$$

$$y'' + \underline{p(t)}\,y' + \underline{q(t)}\,y = \underline{g(t)}$$

$$y'' + \left(\frac{3}{t}\right)y = 1$$

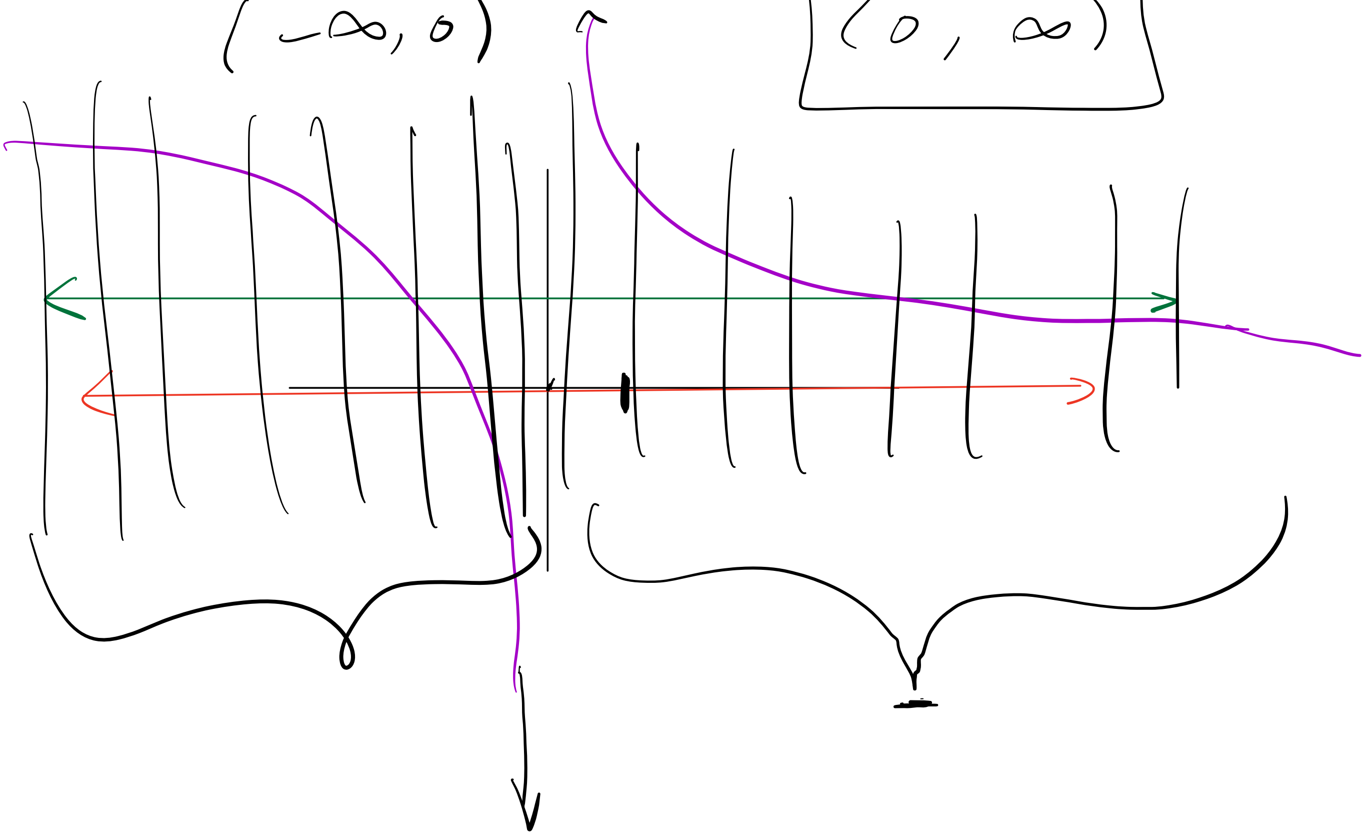
$$= y'' + \underline{0}\,y' + \underline{\left(\frac{3}{t}\right)}\,y = \underline{1}$$

$$p(t) = 0 \rightarrow (-\infty, \infty)$$

$$q(t) = \left(\frac{3}{t}\right) \rightarrow (-\infty, 0)(0, \infty)$$

$- \; g'(t) = 1 \rightarrow (-\infty, \infty)$

$(-\infty, 0)$ $\quad$ $\boxed{(0, \infty)}$

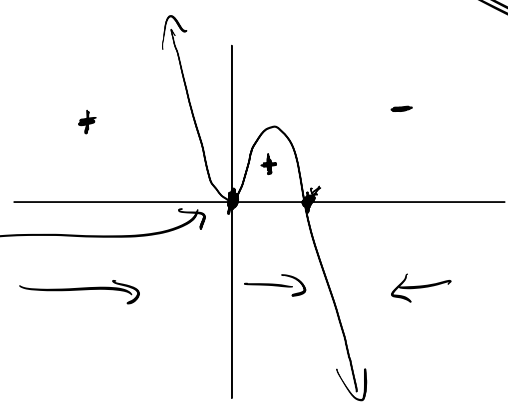


4) Find and classify all equilibrium points.

$y' = y^2(1-y) \Longrightarrow y=0 \quad y=1$

$y^2 - y^3$

$y'' = \boxed{2y - 3y^2}$



Semi-stable

stable

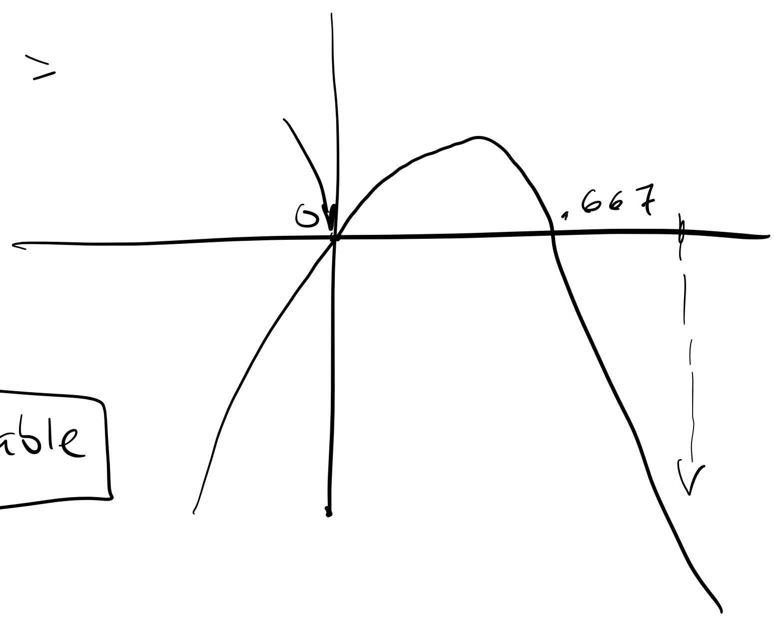
$\longrightarrow \bullet \longleftarrow$

unstable

$\longleftarrow \bullet \longrightarrow$

$y = 1$ = negative

$\boxed{2y - 3y^2} >$



$y = 0, \quad y = 1$

$\boxed{y = 1 \Rightarrow \text{stable}}$

$$\boxed{y = 0 \Rightarrow \text{semi-stable}}$$

5) Write the following 2^{nd}-order differential equations as a system of first-order, linear differential equations.

$$2y'' - 5y' + y = 0 \qquad \begin{array}{l} y(3) = 6 \\ y'(3) = -1 \end{array}$$

$$x_1(t) = y(t)$$
$$x_2(t) = y'(t)$$

$$\boxed{x_1'} = y' = \boxed{x_2}$$

$$\underline{x_2'} = y'' = \underbrace{-\frac{1}{2}y + \frac{5}{2}y'}$$

$$y'' - \frac{5}{2}y' + \frac{1}{2}y = 0$$

$$y'' = \frac{5}{2}y' - \frac{1}{2}y$$

$$-\frac{1}{2}x_1 + \frac{5}{2}x_2$$

$y(3) = 6$
$y'(3) = -1$

$x_1(3) = y(3) = 6$
$x_2(3) = y'(3) = -1$

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{1}{2}x_1 + \frac{5}{2}x_2 \\ &\text{for } x_1(3) = 6, \quad x_2(3) = -1 \end{aligned}}$$

6) Find the critical points and classify the stability.

$$y' = y^3 + 3y^2 + 2y. \qquad \boxed{(1)}$$

$$= y(y^2 + 3y + 2)$$

$$y' = y(y+2)(y+1)$$

$0 \qquad -2 \qquad -1$

$y = 0, \ -2, \ -1$

$y'' = 3y^2 + 6y + 2$ $\boxed{2}$

if > 0, unstable
< 0, stable
$= 0$, we don't know

i) $y = 0 \Rightarrow 3(0)^2 + 6(0) + 2 = 2$

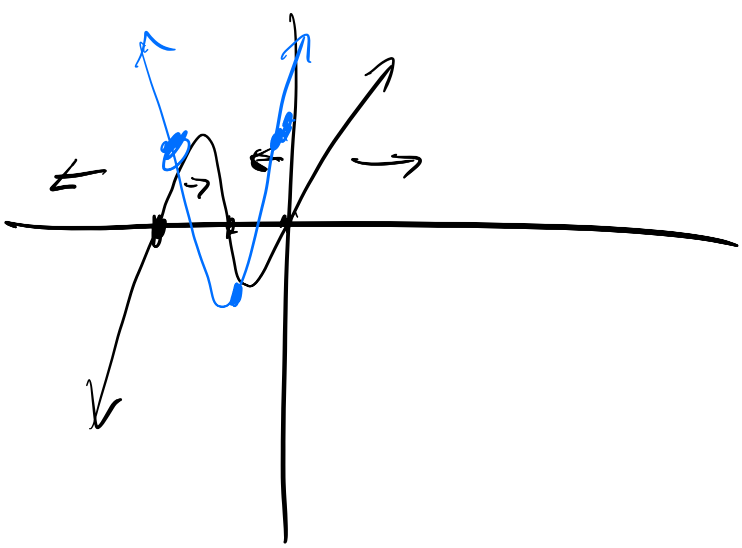
$\boxed{\text{Unstable}}$

ii) $y = -2 \Rightarrow 3(-2)^2 + 6(-2) + 2$

$\boxed{\text{Unstable}}$

iii) $y = -1 \Rightarrow 3(-1)^2 + 6(-1) + 2 = -1$

$\boxed{\text{Stable}}$



a) Solve the general solution and IVP:

$$x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} x, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

A $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ A^*

$$A^* = A - \lambda I \qquad \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$

Solve for λ's by taking $\det(A^*)$.

$ad - bc$

$$(1-\lambda)(2-\lambda) - 6$$

$$2 - 2\lambda - \lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4$$

$$\lambda^2 - 3\lambda - 4$$

$$(\lambda + 1)(\lambda - 4)$$

$$\Longrightarrow \lambda_1 = -1, \quad \lambda_2 = 4$$

These are our eigenvalues.

Eigenvectors : η

$\lambda_1 = -1$ $\qquad\qquad\qquad\qquad\qquad$ $\lambda_2 = 4$

$$A^* = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \qquad\qquad A^* = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2\eta_1 + 2\eta_2 = 0$

$\eta_1 = -1\eta_2$

$\Rightarrow \begin{pmatrix} -1\eta_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\eta_1 = 1$

$-3\eta_1 + 2\eta_2 = 0$

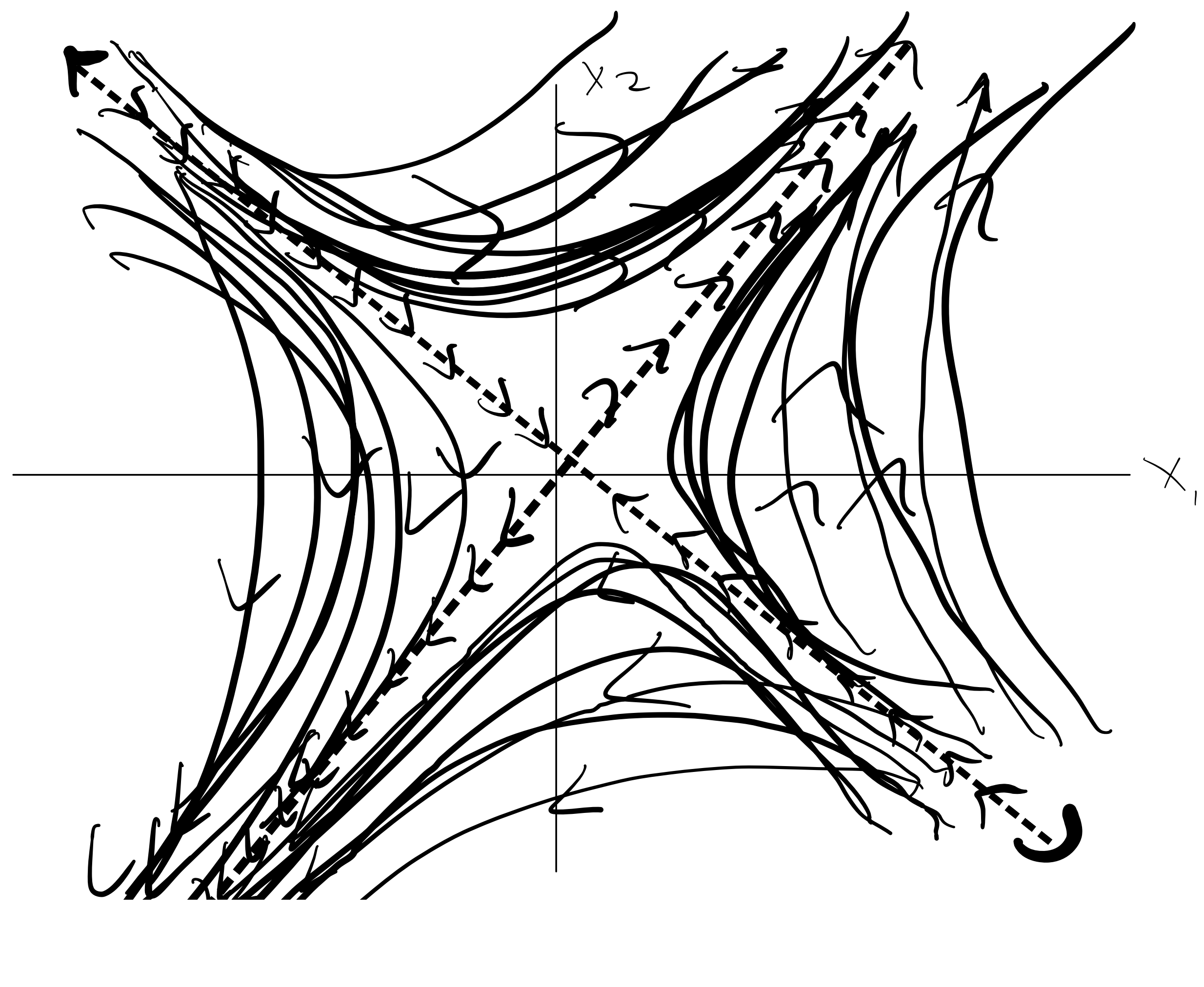
$-3\eta_1 = -2\eta_2 \Rightarrow \eta_1 = \frac{2}{3}\eta_2$

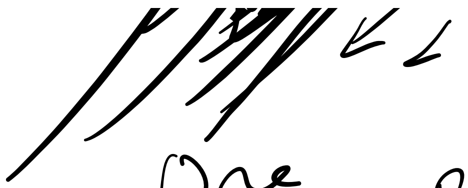
$\begin{pmatrix} 2/3\,\eta_2 \\ \eta_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\eta_2 = 3$

$$x(t) = C_1 e^{\lambda_1 t}\eta_1 + C_2 e^{\lambda_2 t}\eta_2$$

$$= C_1 e^{\boxed{-t}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{\boxed{4}t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$





One positive eigenvalue,
one negative:

$$\boxed{\begin{array}{c}\text{Unstable}\\ \text{saddle pt.}\end{array}}$$

$$x(t) = C_1 e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + C_2 e^{4t}\begin{pmatrix}2\\3\end{pmatrix}$$

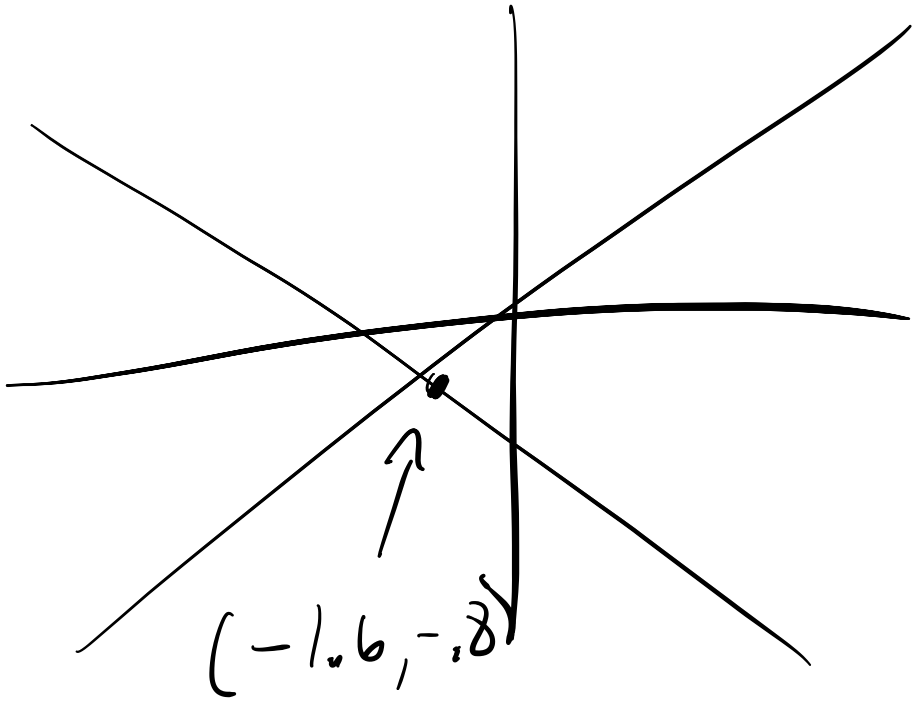
$$\downarrow$$

$$x(0) = \begin{pmatrix}0\\-4\end{pmatrix}$$

$$\begin{pmatrix}0\\-4\end{pmatrix} = C_1 e^{0}\begin{pmatrix}-1\\1\end{pmatrix} + C_2 e^{0}\begin{pmatrix}2\\3\end{pmatrix}$$

$$\begin{pmatrix}0\\-4\end{pmatrix} = C_1\begin{pmatrix}-1\\1\end{pmatrix} + C_2\begin{pmatrix}2\\3\end{pmatrix}$$

$$\left.\begin{aligned} 0 &= -C_1 + 2C_2 \\ -4 &= C_1 + 3C_2 \end{aligned}\right\} \text{Solve the system}$$



$$\boxed{\begin{aligned} C_1 &= -1.6 \\ C_2 &= -.8 \end{aligned}}$$

b) Solve the general solution and IVP:

$$x' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{pmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-3-\lambda) + 36$$

$$-9 + \cancel{3\lambda} \cancel{- 3\lambda} + \lambda^2 + 36$$

$$\lambda^2 + 27 = \boxed{\pm 3\sqrt{3}\, i}$$

$$= 0 \pm 3\sqrt{3}\, i$$

$$a \pm b\, i$$

$a = 0 \Rightarrow$ circle, ellipse

$$A^{*} = \begin{vmatrix} 3 - 3\sqrt{3}\, i & 9 \\ -4 & -3 - 3\sqrt{3}\, i \end{vmatrix}$$

$$\begin{pmatrix} 3 - 3\sqrt{3}\, i & 9 \\ -4 & -3-3\sqrt{3}\, i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(3-3\sqrt{3}\,i)\eta_1 + 9\eta_2 = 0$$

$$9\eta_2 = -(3-3\sqrt{3}\,i)\eta_1$$

$$\eta_2 = -\frac{1}{3}(1-\sqrt{3}\,i)\eta_1$$

$$\eta = \begin{pmatrix} \eta_1 \\ -\frac{1}{3}(1-\sqrt{3}\,i)\eta_1 \end{pmatrix}$$

$$\eta_1 = 3 \Rightarrow \eta_1 = \begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix}$$

$$\eta_2 = \begin{pmatrix} 3 \\ -1-\sqrt{3}\,i \end{pmatrix}$$

$$x_1(t) = \boxed{e^{3\sqrt{3}\,i t}} \begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix}$$

$\downarrow$ Euler's Formula

$$x_1(t) = \left(\cos(3\sqrt{3}t) + i\sin(3\sqrt{3}t)\right)\begin{pmatrix} 3 \\ -1+\sqrt{3}\,i \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 3\cos(3\sqrt{3}t) + 3i\sin(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - i\sin(3\sqrt{3}t) + \sqrt{3}\, i\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} \begin{matrix} 1) \\ 2) \end{matrix}$$

$$x_1(t) = \begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} + i\begin{pmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) + \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix}$$

$$\underbrace{u(t)} + i\,\underbrace{v(t)}$$

$$\vec{x}(t) = C_1 u(t) + C_2 v(t)$$

$$\vec{x}(t) = u(t) + iv(t)$$

$$\boxed{\vec{x}(t) = C_1 \begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} + C_2 \begin{pmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) + \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix}}$$

$$x(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = C_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

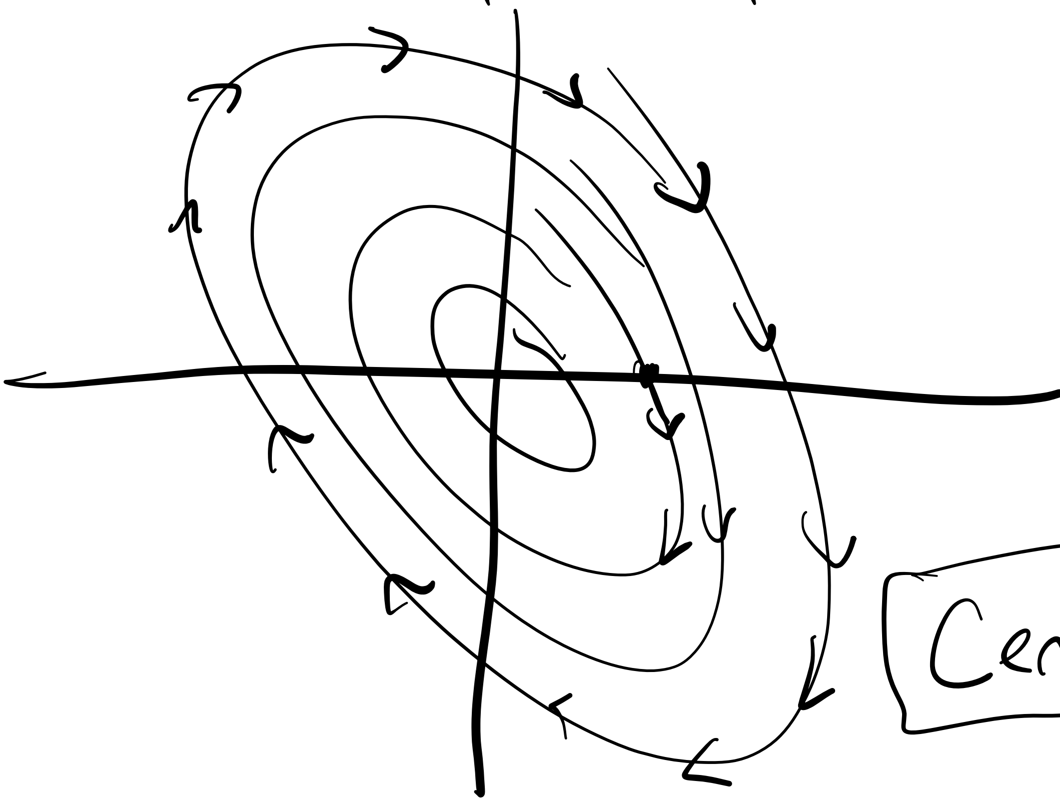
$\therefore \; 3C_1 = 2$ $\qquad \boxed{C_1 = \frac{2}{3}}$

$$\boxed{C_2 = \frac{-10}{3\sqrt{3}}}$$

$$\boxed{\vec{x}(t) = \frac{2}{3} \begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} - \frac{10}{3\sqrt{3}} \begin{pmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) + \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix}}$$

IVP Solution.

Find the phase plane



$\boxed{\text{Center}}$

$\boxed{\text{Neutrally Stable}}$ $\leftarrow$ $a \pm bi$, $a = 0$

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

this tells us direction for a point.

$\begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

c) Solve the general solution and IVP:

$$x' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

(repeated eigenvalues)

$$\det(A - \lambda I) = \begin{pmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix}$$

ad-bc $\quad (7-\lambda)(3-\lambda) + 4$

$$21 - 3\lambda - 7\lambda + 4$$

$$\lambda^2 - 10\lambda + 25$$

$$(\lambda - 5)(\lambda - 5)$$

$$(\lambda - 5)^2 \Rightarrow \lambda_{1,2} = 5$$

Double eigenvalue

$$A^* = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2\eta_1 + 1\eta_2 = 0 \Rightarrow 2\eta_1 = -\eta_2 \Rightarrow \eta_1 = \left(\frac{-1}{2}\right)\eta_2$$

$$\begin{pmatrix} -\frac{1}{2}\eta_1 \\ \eta_2 \end{pmatrix} \quad \eta_2 = -2 \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$x(t) = c_1 e^{\lambda t} \eta + c_2 e^{\lambda t}(\eta t + \vec{p})$$

generalized eigenvector

$$x(t) = c_1 e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{5t}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} t + p\right)$$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$ Generalized eigenvector

$$\begin{cases} 2p_1 + 1p_2 = 1 \\ -4p_1 - 2p_2 = -2 \end{cases}$$

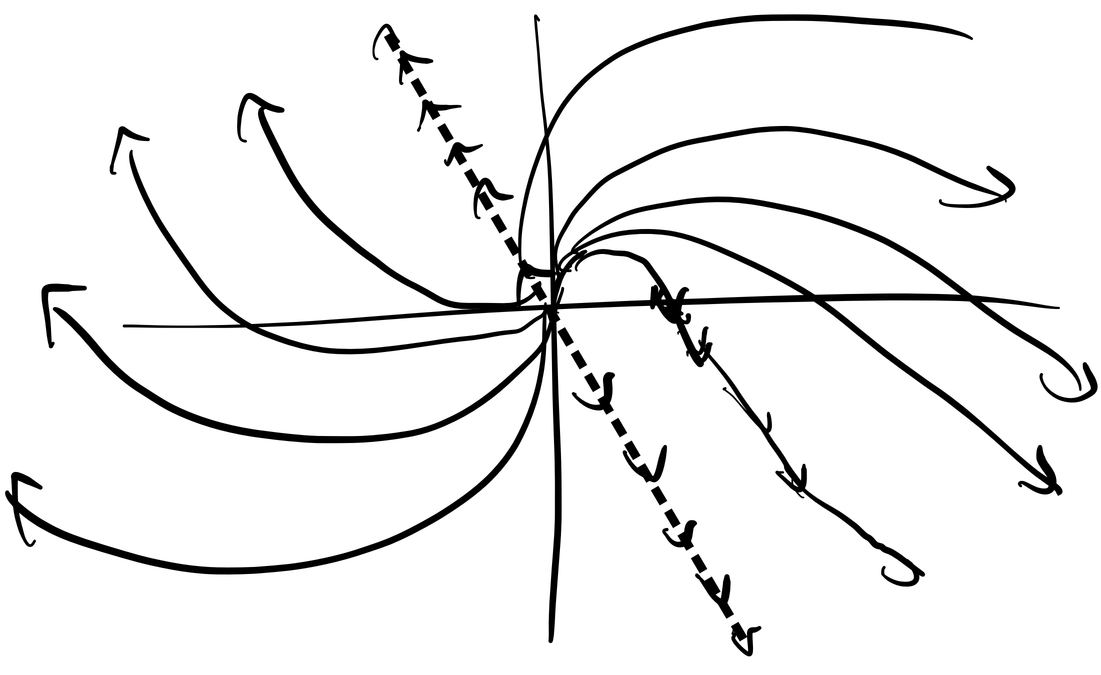
$\boxed{p_2 = 1 - 2p_1}$

G. E. $\begin{pmatrix} p_1 \\ 1-2p_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Set $p_1 = 0$

$$\boxed{x(t) = C_1 e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + C_2 e^{5t}\left(\begin{pmatrix}1\\-2\end{pmatrix}t + \begin{pmatrix}0\\1\end{pmatrix}\right)}$$

Sketch the phase portrait:



degenerate → repeated λ's
unstable → away from 0.

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

$\boxed{\text{improper unstable}}$

IVP.

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} c_1 &= 2 \\ -2c_1 + c_2 &= -5 \end{aligned}\right\} \quad \begin{aligned} &\underline{c_1 = 2} \\ &\underline{c_2 = -1} \end{aligned}$$

IVP:
Solution

$$\boxed{\vec{x}(t) = 2e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \left(te^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}$$