Midterm 2 Review.

2 Cauchy-Euler

1 Set Up U.C.

1 LRC-Series

2 Laplace Transforms

2 Inverse Laplace Transforms

1 Solve U.C. Undetermined Coefficients

1 Solve VoP. Variation of Parameters

$$ax^2y'' + bxy' + cy = 0$$

1) Solve $x^2y'' - 9xy' = 0, \; x > 0.$

$t = \ln(x).$

$a = 1 \qquad b = -9 \qquad c = 0.$

$$\Downarrow$$

$$aY''(t) + (b-a)Y'(t) + 0Y = 0.$$

$$Y''(t) - 10Y'(t) = 0.$$

$$\Downarrow$$

$$\lambda^2 - 10\lambda = 0.$$

$$\lambda(\lambda - 10) = 0.$$

$$\lambda = 0, \; 10.$$

$$y_c = C_1 e^{0t} + C_2 e^{10t}.$$

$$\Downarrow$$

$$\boxed{\begin{aligned} x &= e^{t} \\ t &= \ln(x) \end{aligned}}$$

$$y_c = C_1 + C_2 e^{10t}$$

$$\Downarrow$$

$$\boxed{y_c = C_1 + C_2 x^{10}}$$

2) Solve

$$x^2 y'' - 9xy' = 0,$$

$y(1) = 1.$

$y'(1) = 10.$

$$y_c(x) = C_1 + C_2 x^{10}.$$

$\Downarrow$

$$y'_c(x) = 10 C_2 x^9$$

$$C_1 + C_2 = 1.$$

$$10 C_2 = 10$$

$$C_2 = 1$$

$$C_1 + 1 = 1$$

$$C_1 = 0$$

$$\boxed{y = x^{10}}$$

3) Given a differential equation, determine a suitable form for $y_p(t)$ if using Method of Underdetermined Coefficients. Do not solve.

$$y'' + P(t)y' + H(t)y = 2t - 3 + 8te^{4t}$$

$$y_c(t) = C_1 e^{6t} + C_2 e^{-3t}$$

$$\underbrace{2t-3} + \underbrace{8te^{4t}}$$

$$y_p(t) = \boxed{(At+B) + e^{4t}(C+Dt)}$$

4) LRC- Series.

Obtain $q(t)$.

$L = 1$

$R = 2$

$C = \frac{1}{8}$

$E(t) = 2\cos(t)$.

$$\left[Lq'' + Rq' + \frac{q}{C} = E(t) \right]$$

$$\boxed{q'' + 2q' + 8q = 2\cos(t)}$$

5) Laplace Transform.

Find the Laplace Transform of the solution to

$$y'' + 8y' + 2y = 0.$$

$$y(0) = 1.$$

$$y'(0) = 0.$$

$$\mathcal{L}(y'') + \mathcal{L}(8y') + \mathcal{L}(2y) = \mathcal{L}(0)$$

$$= \mathcal{L}(y'') + 8\mathcal{L}(y') + 2\mathcal{L}(y) = 0$$

$$s^2 Y(s) - s y(0) - y'(0)$$
$$+ 8s Y(s) - 8y(0)$$
$$+ 2 Y(s)$$
$$= 0.$$

$$= Y(s)\left(s^2 + 8s + 2\right) - s - 8 = 0$$

$$Y(s)(s^2 + 8s + 2) = s + 8$$

$$\boxed{Y(s) = \frac{s+8}{s^2 + 8s + 2}}$$

6) Find the inverse Laplace Transform of $\frac{s+9}{s^2+6s+5}$.

$$\frac{s+9}{s^2+6s+5} = \frac{s+9}{(s+1)(s+5)}.$$

PFD: $$\frac{s+9}{(s+1)(s+5)} = \frac{A}{s+1} + \frac{B}{s+5}.$$

$$s+9 = A(s+5) + B(s+1).$$

$$s+9 = As + 5A + Bs + B$$

$$s+9 = s\underbrace{(A+B)}_{1} + \underbrace{5A+B}_{9}$$

$$\begin{cases} A+B=1 \\ 5A+B=9 \end{cases}$$

$$A=1-B$$

$$5(1-B)+B=9$$

$$5-5B+B=9$$

$$-4B=4$$

$$\underline{B=-1}$$

$$A=1-(-1)$$

$$\underline{A=2}$$

11

$$\Downarrow$$

$$A = 2, \quad B = -1.$$

$$\frac{s+9}{(s+1)(s+5)} = \frac{2}{s+1} + \frac{-1}{s+5}$$

$$= 2\frac{1}{s+1} - 1\frac{1}{s+5}.$$

So, $y(t) =$

$$\mathcal{L}^{-1}\left\{2\frac{1}{s+1} - 1\frac{1}{s+5}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 1\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$\boxed{= 2e^{-t} - e^{-5t}}$$

7) Find the Laplace Transform of $u_\pi(t)\sin(t)$.

$$\mathcal{L}\{u_\pi \sin(t)\} = e^{-\pi s}\mathcal{L}\{\sin(t)+\pi\}.$$

(section 5.5, Eqn 6).

$$\boxed{\mathcal{L}(u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}}$$

$$\Downarrow$$

$$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t)+c\}$$

$\sin(t)+\pi = -\sin(t)$

$$= -e^{-\pi s}\mathcal{L}\{\sin(t)\}.$$

$$= -e^{-\pi s}\frac{1}{s^2+1}$$

$$\boxed{= -\frac{e^{-\pi s}}{s^2+1}}$$

8) Find the inverse Laplace Transform of

$$\frac{x+4}{s^2+8s+32}.$$

$$ax^2+bx+c=0 \Longrightarrow \boxed{a(x+d)^2+e=0.}$$

$$d=\frac{b}{2a}$$

$$e=c-\frac{b^2}{4a}$$

$$\frac{x+4}{(x+4)^2+16}$$

$$\|$$

$$\Downarrow$$

$$= \mathcal{L}^{-1}\left\{ \frac{(x+4)}{(x+4)^2+16} \right\}$$

$$\frac{(s-a)}{(s-a)^2+b^2},\ s>a \Rightarrow e^{at}\cos(bt)$$

$$a=-4,\quad b=4$$

$$\Downarrow$$

$$= \boxed{e^{-4t}\cos(4t)}$$

9) Find the particular solution using method of underdetermined coefficients.

$$8y'' - 2y' - 6y = \sin(4t).$$

$$y_c(t) = C_1 e^{3t} + C_2 e^{-4t}.$$

$$y_p(t) = A\cos(4t) + B\sin(4t)$$

$$y'_p(t) = -4A\sin(4t) + 4B\cos(4t)$$

$$y''_p(t) = -16A\cos(4t) - 16B\sin(4t)$$

$$\Downarrow$$

$$8\left(-16A\cos(4t) - 16B\sin(4t)\right)$$
$$-2\left(-4A\sin(4t) + 4B\cos(4t)\right)$$
$$-6\left(A\cos(4t) + B\sin(4t)\right)$$

$$= -128A\cos(4t) - 128B\sin(4t)$$
$$+ 8A\sin(4t) - 8B\cos(4t)$$
$$- 6A\cos(4t) - 6B\sin(4t)$$

$$= -134A\cos(4t) - 8B\cos(4t)$$
$$+ 8A\sin(4t) - 134B\sin(4t)$$

$$= (8A - 134B)\sin(4t) + (-134A - 8B)\cos(4t)$$

$$(134)^2 + (8)^2$$

$$A = -\frac{134}{18020} \qquad B = -\frac{8}{18020}$$

$$\boxed{y_p(t) = \frac{-134}{18020}\cos(4t) - \frac{8}{18020}\sin(4t)}$$

10) (Review) Find the particular solution of $y'' + y = \sec(t)$.

Lec 12
4.7
notes

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

$$y_c = C_1 \underbrace{\cos(t)}_{y_1} + C_2 \underbrace{\sin(t)}_{y_2}$$

$$X_p = \Phi \int \Phi^{-1} g(t)\, dt$$

$$\Phi = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$ad-bc : |A|$$

$$\phi = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

$$|\phi| = 1$$

$$\phi^{-1} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

$$g = \begin{bmatrix} 0 \\ \sec(t) \end{bmatrix}$$

$$\phi^{-1} g = \begin{bmatrix} -\sin(t)\sec(t) \\ 1 \end{bmatrix}$$

$$\int \phi^{-1} g \, dt = \begin{bmatrix} \ln\left|\cos(t)\right| \\ t \end{bmatrix}$$

$$\Phi \int \Phi^{-1} g \, dt = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \ln|\cos(t)| \\ t \end{bmatrix}$$

$$= \begin{bmatrix} \cos t \ln|\cos t| + t \sin t \\ -\sin(t) \ln|\cos t| + t \cos t \end{bmatrix} \begin{array}{l} = u_1(t) \\ = u_2(t) \end{array}$$

$$\boxed{y = C_1 \cos(t) + C_2 \sin(t) + \cos t \ln|\cos t| + t \sin(t)}$$

11) E.g. VoP: Find the particular solution:

$$y'' + 4y = \sec^2(2t).$$

$$\longrightarrow y_c(t) = C_1\cos(2t) + C_2\sin(2t).$$

Solve, and then:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} A\ F(t) \\ B\ G(t) \end{pmatrix}$$

$$\left[\begin{array}{l} y_p(t) = AF(t)\cos(2t) + BG(t)\sin(2t). \\ \qquad u_1(t)\cos(2t) + u_2(t)\sin(2t) \end{array}\right]$$