Quiz #1 Review

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1) Separation of variables
2) Integrating factor
3) Growth story problem

1) $\dfrac{dy}{dx} = \dfrac{(e^y+2)^2 e^{-y}}{(e^x+1)^4 e^{-x}}$

$$dy\,(e^x+1)^4 e^{-x} = dx\,(e^y+2)^2 e^{-y}$$

$$dy\,\frac{1}{e^{-y}(e^y+2)^2} = dx\,\frac{1}{(e^x+1)^4 e^{-x}}$$

$$dy\,\frac{e^y}{(e^y+2)^2} = dx\,\frac{e^x}{(e^x+1)^4}$$

$$\int dy\,\frac{e^y}{(e^y+2)^2} = \int dx\,\frac{e^x}{(e^x+1)^4}$$

$$\left|\qquad\right| \qquad\qquad \left|\qquad\right|$$

↓

substitution

$u = e^y + 2$

$\frac{du}{dy} = e^y$

$dy = \frac{du}{e^y}$

$dy = e^{-y}\,du$

$\int dy\ \frac{e^y}{(e^y+2)^2}$

$= \int e^{-y}\,\frac{e^y}{u^2}\,du$

$= \int du\ \frac{1}{u^2}$

$\int u^n = \frac{u^{n+1}}{n+1}$ | power rule

↓

substitution

$u = e^x + 1$

$\frac{du}{dx} = e^x$

$dx = e^{-x}\,du$

$\int dx\ \frac{e^x}{(e^x+1)^4}$

$= \int e^{-x}\,\frac{e^x}{u^4}\,du$

$= \int du\ \frac{1}{u^4}$

$\int u^n\,du = \frac{u^{n+1}}{n+1}$

$$= -\frac{1}{u}$$

$$= -\frac{1}{e^y+2} + C$$

$$= -\frac{1}{3u^3}$$

$$= -\frac{1}{3(e^x+1)^3} + C$$

$$\frac{1}{e^y+2} = \frac{1}{3(e^x+1)^3} + C$$

2)

$y' =$ something $\Rightarrow$? y

a) I.F.

$$\frac{dy}{dx} = 4y + 8x + 2$$

$$\frac{dy}{dx} - 4y = 8x + 2 \qquad (1)$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$\mu(x)$

$$\text{I.F.} = e^{\int P(x)\,dx}$$

$$\frac{dy}{dx} - 4y = 8x + 2$$

I.F.: $e^{\int -4\,dx} = e^{-4x}$

$$\frac{dy}{dx} - 4y = 8x + 2 \qquad (1)$$

$$e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = e^{-4x}(8x+2)$$

$$e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = 8xe^{-4x} + 2e^{-4x}$$

$* \left(e^{-4x}y\right)'$

$$e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = 8xe^{-4x} + 2e^{-4x}$$

$$\left(e^{-4x}y\right)' = \overset{(1)}{8}\times e^{-4x} \overset{(2)}{+2e^{-4x}}$$

$$\int\left(e^{-4x}y\right)' = \int 8\times e^{-4x} + 2e^{-4x}\,dx$$

$$e^{-4x}y = \int 2\left(4\times e^{-4x} + e^{-4x}\right)dx$$

$$= 2\int (4x+1)e^{-4x}\,dx$$

$$\int fg' = fg - \int f'g$$

$f = 4x+1 \qquad g' = e^{-4x}$

$f' = 4 \qquad g = -\frac{e^{-4x}}{4}$

$$= 2\left[-\frac{(4x+1)e^{-4x}}{4} - \int -e^{-4x}\,dx \right]$$

$$\int -e^{-4x}\,dx = \frac{e^{-4x}}{4}$$

$$= 2\left[-\frac{(4x+1)e^{-4x}}{4} - \frac{e^{-4x}}{4} + C \right]$$

$$= -\frac{(4x+1)e^{-4x}}{2} - \frac{e^{-4x}}{2} + C$$

$$= -e^{-4x}\left(2x + \frac{1}{2} + \frac{1}{2}\right) + C$$

$$= -(2x+1)e^{-4x} + C$$

$$e^{-4x} y = -(2x+1)e^{-4x} + C$$

$$\boxed{y = -(2x+1) + Ce^{4x}}$$

This is our general solution.

What if we are given an initial value, say $y(0) = 2$?

Now, we can solve for the particular solution.

$$\boxed{y = -(2x+1) + Ce^{4x}}$$

$$y(0) = -(2(0)+1) + Ce^{4(0)}$$

$$2 = -(1) + Ce^{0}$$

$$2 = -1 + C$$

$$3 = C$$

$$\boxed{y = -(2x+1) + 3e^{4x}}$$

This is our particular solution.

3) A tank contains 40 gal water with 5 lbs salt in solution. Water containing $\frac{1}{10}$ lbs of salt per gallon is entering at a rate of 2 gal/min, and the well-stirred solution in the tank is leaving at the same rate.

Wksht 3, 2.3 #1

a) Write down the differential equation for Q(t), the amount of salt in the tank.

pg. 11 in textbook

$$\frac{dQ}{dt} = \underbrace{c_i(t)\, r_i(t)}_{\text{rate in}} - \underbrace{\left\{\frac{Q(t)}{V(t)}\right\} r_o(t)}_{\text{rate out}}$$

$c_i(t) = \frac{1}{10}$, $r_i(t) = 2$, $V(t) = 40$, $r_o(t) = 2$

$$\frac{dQ}{dt} = \left(\frac{1}{10}\right)(2) - \left(\frac{Q(t)}{40}\right)(2)$$

$$= \frac{1}{5} - \frac{Q}{20}$$

b) Write the initial value problem for $Q(t)$.

$$Q(0) = 5$$

c) Find $Q(t)$ (solve the initial value problem).

Remember:

1) $\frac{dT}{dt} = -k(T - T_0)$

$T = T_0 + Ce^{-kt}$

} Newton's Law of cooling

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2) $\frac{dT}{dt} = -kT$

$T = Ce^{-kt}$

} Exponential growth and decay

We have 1).

$\frac{dT}{dt} = -k(T - T_0)$ or $\frac{dT}{dt} = k(T_0 - T)$

$\frac{dQ}{dt} = \frac{1}{5} - \frac{Q}{20}$

$$\frac{dQ}{dt} = \frac{1}{20}(4 - Q)$$

$$T = T_0 + Ce^{-kt}$$

$$Q = 4 + Ce^{-\frac{1}{20}t}$$

$Q(0) = 5$

$$5 = 4 + Ce^{0}$$

$$C = 1$$

$$Q(t) = 4 + e^{-\frac{1}{20}t}$$