1.

$$X' = \begin{bmatrix} 3 & -2 \\ 3 & 10 \end{bmatrix} X.$$

A) Find the general solution.

We need the eigenvalues and eigenvectors.

$$\begin{bmatrix} 3 & -2 \\ 3 & 10 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3-\lambda & -2 \\ 3 & 10-\lambda \end{bmatrix}$$

Then, find the determinant.

ad- bc.

$$(3-\lambda)(10-\lambda) - (-2)(3)$$

$$= 30-10\lambda-3\lambda+\lambda^2 + 6$$

$= 30 - 13\lambda + \lambda^2 + 6$

$= \lambda^2 - 13\lambda + 36$

$= (\lambda - 9)(\lambda - 4)$

$\lambda_1 = 9 \; ; \; \lambda_2 = 4.$

These are our eigenvalues.

$\lambda_1 = 9$

$$\begin{bmatrix} 3-9 & -2 \\ 3 & 10-9 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -2 \\ 3 & 1 \end{bmatrix}$$

$||$

$\lambda_2 = 4$

$$\begin{bmatrix} 3-4 & -2 \\ 3 & 10-4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 3 & 6 \end{bmatrix}$$

$||$

Our eigenvector is $\begin{pmatrix} b \\ -a \end{pmatrix}$, or $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Our eigenvector is $\begin{pmatrix} b \\ -a \end{pmatrix}$, or $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Now, we have

eigenvalue 1: 9

eigenvector 1: $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

eigenvalue 2: 4

eigenvector 2: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Our general solution can be written as:

$$X(t) = C_1 e^{\lambda t_1} \eta_1 + C_2 e^{\lambda t_2} \eta_2,$$

where $\eta_{1,2}$ are our eigenvectors.

So, we have

$$\boxed{x(t) = C_1 e^{9t}\begin{pmatrix}-1\\3\end{pmatrix} + C_2 e^{4t}\begin{pmatrix}-2\\1\end{pmatrix}.}$$

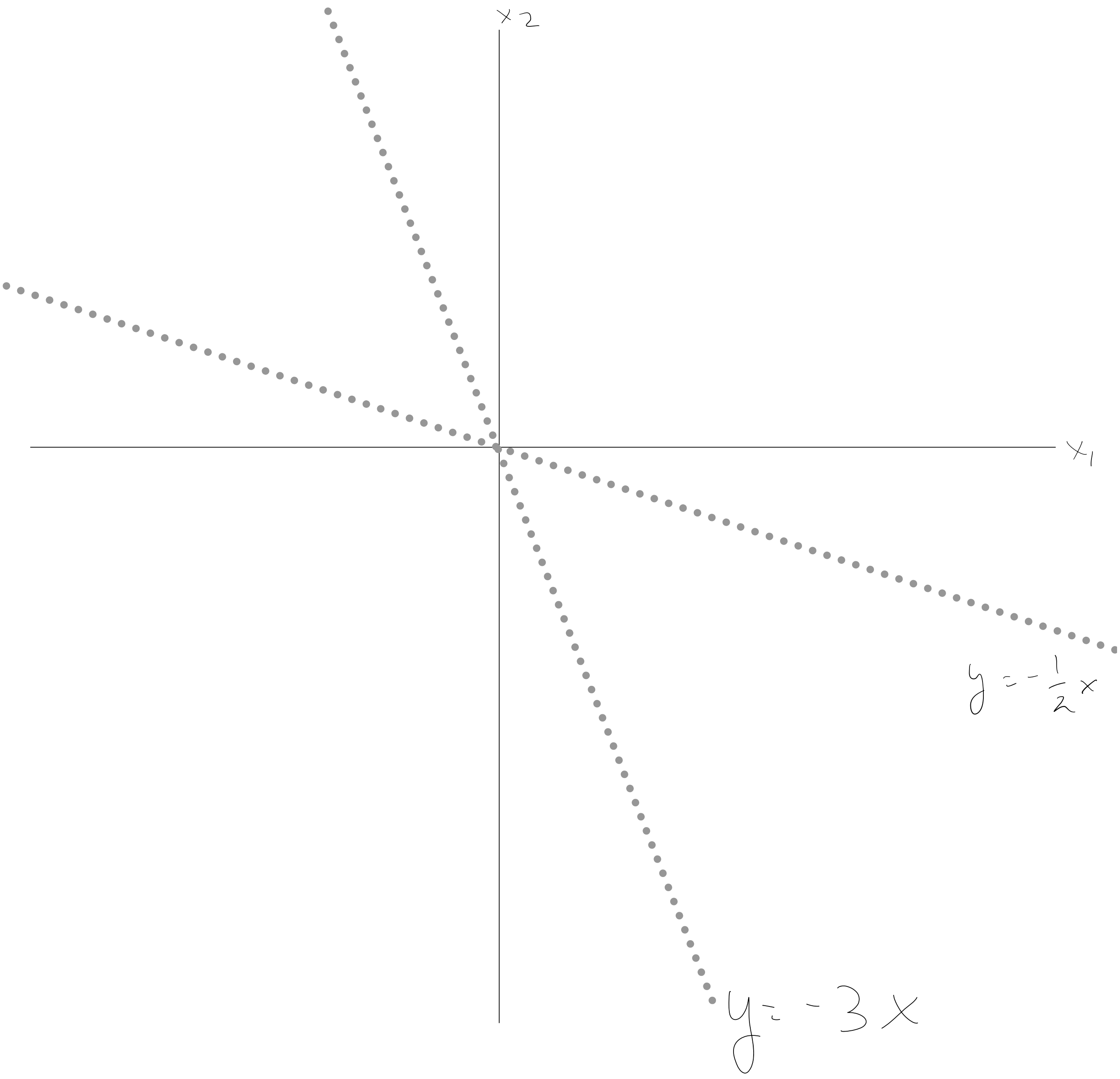
1B) Sketch the phase portrait.

Our eigenvalues tell us the lines:

$$\begin{pmatrix}-1\\3\end{pmatrix} \Longrightarrow y = \left(\frac{3}{-1}\right)x = -3x$$

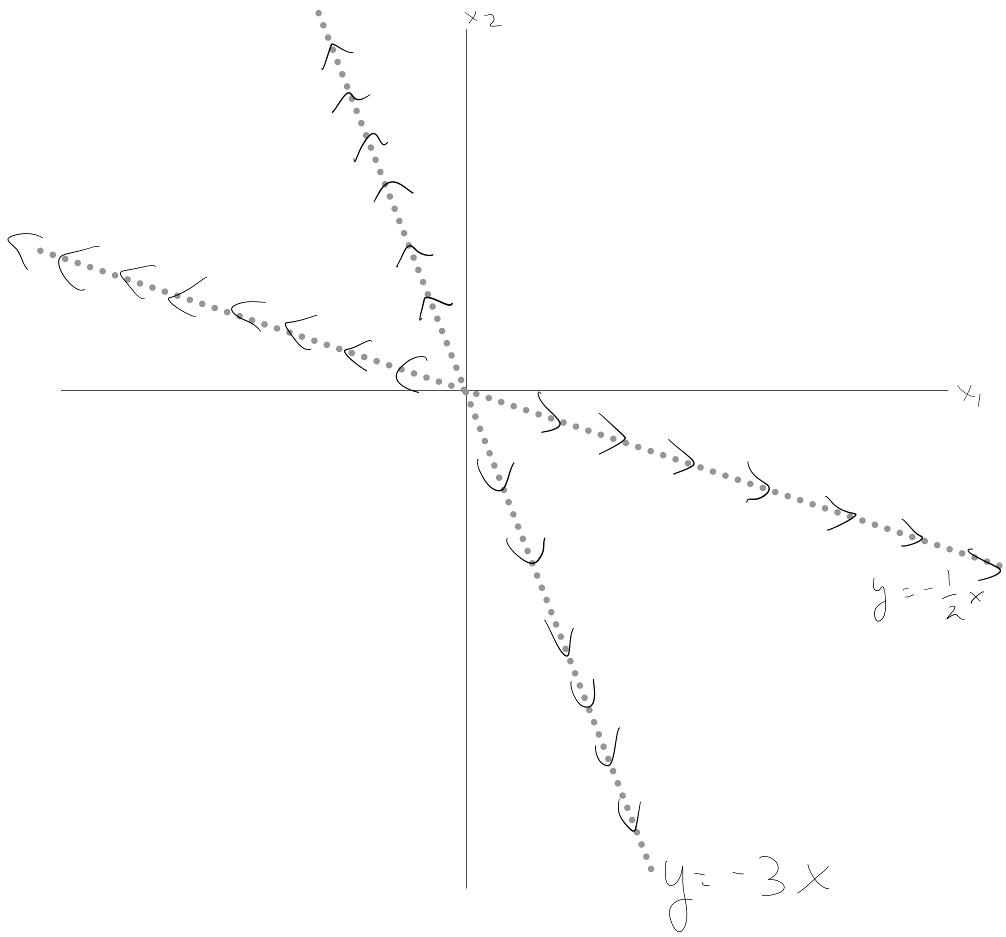
$$\begin{pmatrix}-2\\1\end{pmatrix} \Longrightarrow y = \left(\frac{1}{-2}\right)x = -\frac{1}{2}x$$

Let's plot them!

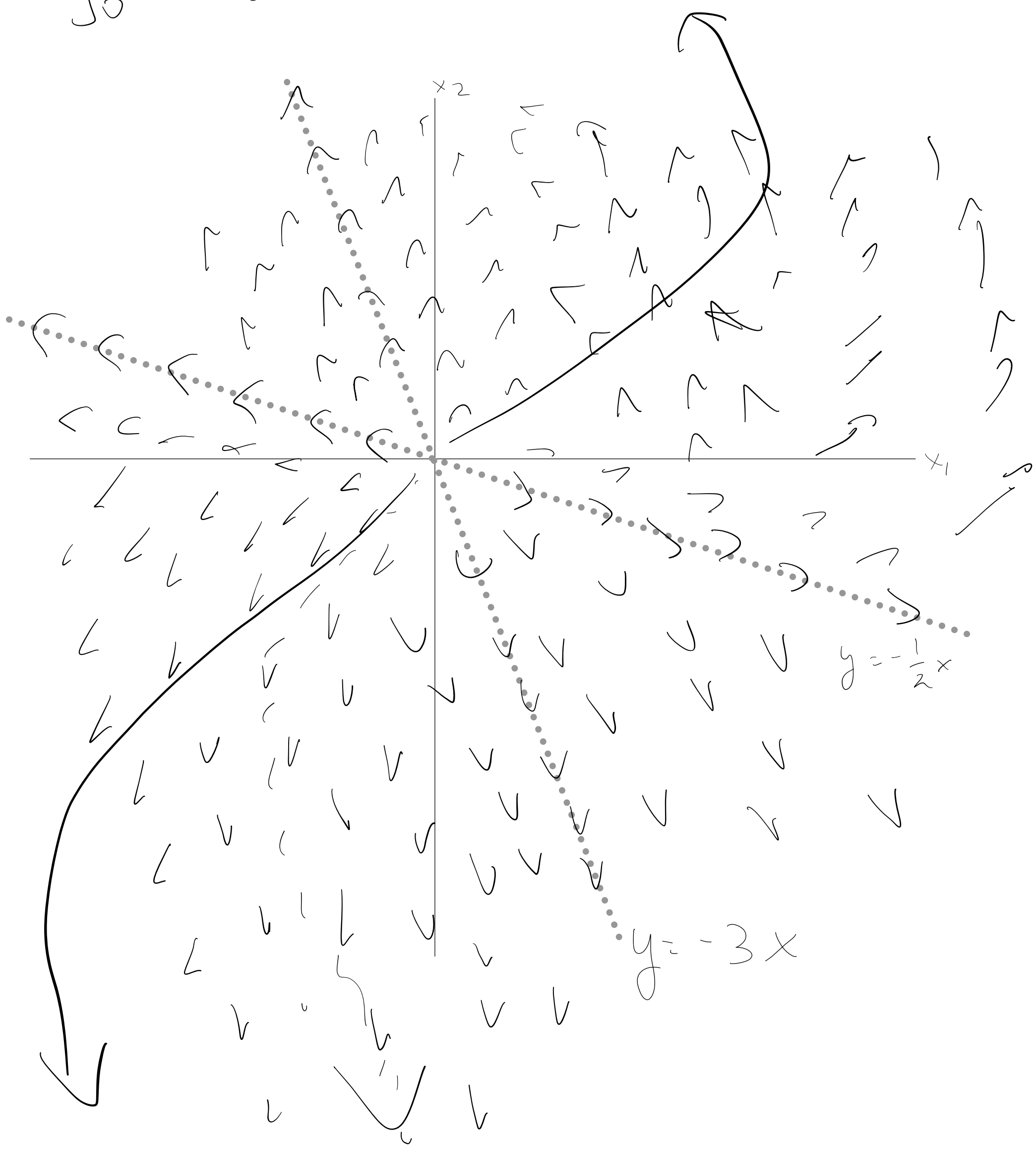


x_2
x_1
$y = -\frac{1}{2}x$
$y = -3x$

Also, note positive eigenvalues move <u>away</u> from the origin, and negative eigenvalues move toward the origin. Here, both our eigenvalues are positive:



Also, it is moving faster toward the line associated with the greater eigenvalue (eigenvalue $= 9$, eigenvector $= \begin{pmatrix} -1 \\ 3 \end{pmatrix}$), so we can draw that in:



1c) Solve the IVP.

$x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

$$x(t) = C_1 e^{9t} \begin{pmatrix} -1 \\ 3 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 e^{9(0)} \begin{pmatrix} -1 \\ 3 \end{pmatrix} + C_2 e^{0} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

(1) $3 = -C_1 - 2C_2$

(2) $2 = 3C_1 + C_2 \Rightarrow C_2 = 2 - 3C_1$

$$3 = -C_1 - 2(2 - 3C_1)$$

$$3 = -C_1 - 4 + 6C_1$$

$$5C_1 - 4 = 3$$

$$5C_1 = 7$$

$$\boxed{C_1 = \frac{7}{5}}$$

Plug into (2):

$$2 = 3\left(\frac{7}{5}\right) + C_2$$

$$2 = \frac{21}{5} + C_2$$

$$C_2 = 2 - \frac{21}{5}$$

$$C_2 = \frac{10}{5} - \frac{21}{5}$$

$$\boxed{C_2 = -\frac{11}{5}}$$

$$\boxed{x(t) = \frac{7}{5} e^{9t} \begin{pmatrix} -1 \\ 3 \end{pmatrix} - \frac{11}{5} e^{4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.}$$

1d) What is its stability?

$\boxed{\text{Unstable}}$, both eigenvalues are positive.

☆ Note, for eigenvalues:

2 Real Positive = unstable node

2 Real Negative = stable node

Mixed pos/neg real = unstable saddle point

2) $y' = y^3 + 3y^2 + 2y$

Find the critical points and classify the stability.

(1) $y' = y^3 + 3y^2 + 2y$

$= y(y^2 + 3y + 2)$

$= y(y+2)(y+1)$

$\Rightarrow \quad y = 0, \quad y = -2, \quad y = -1$

Now, take the derivative of (1):

(2) $= 3y^2 + 6y + 2$

We can plug our fixed points into (2).

If > 0, it is unstable.

< 0, it is stable.

$= 0$, it requires further analysis.

i) $y=0 \Rightarrow 3(0)^2+6(0)+2 = 2$ unstable

ii) $y=-2 \Rightarrow 3(-2)^2+6(-2)+2 = 2$ unstable

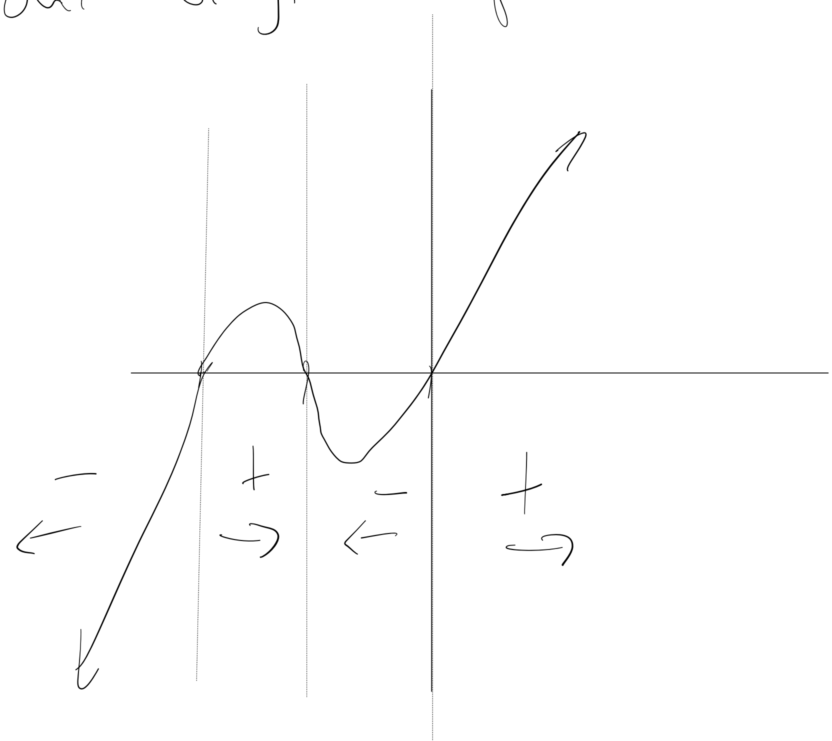
iii) $y=-1 \Rightarrow 3(-1)^2+6(-1)+2 = -1$ stable

This makes sense, bc look at the graph for $y' = y^3+3y^2+2y$

(our original problem):

$- = \leftarrow$

$+ = \rightarrow$



$\leftarrow \quad \rightarrow \quad =$ unstable.

$\rightarrow \quad \leftarrow \quad =$ stable.