Math 2552 Quiz 5 Review

November 27 2023

1 Inverse Laplace with Convolution

Find the inverse laplace transform of the following expression using convolution:

$$\frac{1}{(s+5)^6(s^2+9)}$$

Solution.

Consider that

$$\frac{1}{(s+5)^6(s^2+9)} = \frac{1}{(s+5)^6}\frac{1}{(s^2+9)} = F(s)G(s),$$

where

$$F(s) = \frac{1}{(s+5)^6} \text{ and } G(s) = \frac{1}{(s^2+9)}.$$

Note that we have the following two inverse laplace transform rules:

$$H(s) = \frac{n!}{(s-a)^{n+1}} \to h(t) = t^n e^{at} \tag{1}$$

and

$$H(s) = \frac{a}{s^2+a^2} \to h(t) = \sin(at) \tag{2}$$

You should know how to obtain inverse laplace transforms from a combination of transforms on your formula sheet. For example, (1) is not on your formula sheet, but it is a combination of two transforms on your formula sheet.

We want to put $F(s)$ and $G(s)$ into an appropriate form, so that we use rules (1) and (2). Note that if we multiply $F(s)$ by $\frac{5!}{5!}$, we can write:

$$F(s) = \frac{1}{5!}\frac{5!}{(s+5)^6}.$$

We may also multiply $G(s)$ by $\frac{3}{3}$, and we have

$$G(s) = \frac{1}{3}\frac{3}{(s^2+9)}.$$

We may now use rules (1) and (2), so that we have

$$f(t) = \frac{1}{5!}t^5 e^{-5t}$$

and

$$g(t) = \frac{1}{3}\sin(3t).$$

Recall the convolution equation:

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{3}$$

We may now substitute $f(t)$ and $g(t)$ into equation (1). First, we move our coefficients outside of the integral (by the property of linearity).

$$\frac{1}{5!}\frac{1}{3} = \frac{1}{360},$$

and we have

$$\frac{1}{360}\int_0^t (t-\tau)^5 e^{-5(t-\tau)} \sin(3\tau)d\tau$$

.

We do not need to solve the integral analytically, and so we are done.

2 Critical Points of Almost Linear Systems

Find the critical points of the following system:

$$x' = x(3 - x - 2y) \tag{1}$$

$$y' = y(4 - 2y - 4x). \tag{2}$$

Solution.

Note that there are two terms in (2): "y" and "$4 - 2y - 4x$". We can set both terms equal to 0 to find our solutions.

First, let us consider the first term in (2), for which we have $y = 0$. Let us substitute $y = 0$ into (1), and we solve:

$$x(3 - x - 2(0)) = 0$$
$$x(3 - x) = 0$$
$$x = 0, x = 3.$$

We have $x = 0$, $x = 3$, and $y = 0$. We can now construct combinations of these solutions to identify two fixed points: $(0, 0)$ and $(3, 0)$.

Second, we will consider the other term in (2): $4 - 2y - 4x$. Here, a solution for y is not a single real value but is instead a function of x. We may write

$$4 - 2y - 4x = 0$$
$$4 - 4x = 2y$$
$$2 - 2x = y.$$

We may now substitute y into (1) to solve for x, and we have

$$x(3 - x - 2y) = 0$$
$$x(3 - x - 2(2 - 2x)) = 0$$
$$x(3 - x - 4 + 4x) = 0$$
$$x(-1 + 3x) = 0$$
$$x = 0, x = \frac{1}{3}.$$

We may substitute each solution of x into the second term of (2) to find the corresponding y-value:

$$\begin{aligned} 4 - 2y - 4(0) &= 0 \\ 4 - 2y &= 0 \\ 4 &= 2y \\ y &= 2, \end{aligned}$$

and

$$\begin{aligned} 4 - 2y - 4(\frac{1}{3}) &= 0 \\ 4 - 2y - \frac{4}{3} &= 0 \\ \frac{8}{3} - 2y &= 0 \\ y &= \frac{4}{3}. \end{aligned}$$

Therefore, we have the following two critical points: (0,2) and $(\frac{1}{3}, \frac{4}{3})$.

In total, we have identified four critical points:

$$\begin{gathered} (0,0) \\ (3,0) \\ (0,2) \\ (\frac{1}{3}, \frac{4}{3}), \end{gathered}$$

and we are done.

3 Approximate Linear Systems

Write the approximate linear system near (2,0) for the following nonlinear system:

$$\begin{aligned} x' &= x(2 - 4x - 0.5y) \\ y' &= y(1 - 3y - 0.75x) \end{aligned}$$

Solution.

First, we can rewrite x' and y':

$$x' = 2x - 4x^2 - 0.5xy$$

$$y' = y - 3y^2 - 0.75xy$$

Then, we compute the Jacobian matrix J of this system:

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 2 - 8x - 0.5y & -0.5x \\ -0.75y & 1 - 6y - 0.75x \end{pmatrix}.$$

Now, we evaluate at (2,0):

$$\begin{pmatrix} 2 - 8(2) - 0.5(0) & -0.5(2) \\ -0.75(0) & 1 - 6(0) - 0.75(2) \end{pmatrix} = \begin{pmatrix} -14 & -1 \\ 0 & -0.5 \end{pmatrix}.$$

We may now write our solution:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -14 & -1 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} x-2 \\ y \end{pmatrix}.$$

Our solution, in alternative notation you may have seen, can be written:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -14 & -1 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$

where we define $a = x - 2$ and $b = y$.

4 Stability of Linearized System

Determine the stability of the point (2,0)[1]

Solution.

Recall from the solution from the previous question is

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -14 & -1 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

We may now solve for the eigenvalues of our matrix. Remember that to solve for the eigenvalues for a matrix A, we first compute A - λI and then solve for the roots of a polynomial $ad - bc$.

For a 2 x 2 matrix, when either b or c is 0, our eigenvalues are equal to the diagonal elements (this is because $bc = 0$ so $ad - bc = ad - 0 = ad$. The roots of a and d can be found independently, and are equal to the diagonal).

Therefore, our eigenvalues are -14 and -0.5. Since our eigenvalues are real, different, and negative, the system is **asymptotically stable**. If our eigenvalues were real, different, and positive, the system would be unstable. If both were real (but one is positive and one is negative), the system would be an unstable saddle point.

5 Additional Comments

As we've seen, nonlinear differential systems are difficult and often impossible to solve analytically, requiring alternative methods. Remember from calculus that we can approximate the solution to any differential system arbitrarily well, given an arbitrarily small step size in our numerical integration technique. For example, there is no closed-form solution for the cumulative density function of a Gaussian. It depends on the error function, which can only be computed using numerical techniques. Similarly, in differential equations, we sometimes can only use numerical techniques to approximate our solution.

In our case, we are linearizing around an operating point. When our operating point is a fixed point (and we do not have a 0 eigenvalue), we are mathematically guaranteed to observe the same qualitative behavior as the nonlinear system via the Hartman-Grobman theorem. Linearization is an effective tool for teaching us about some things about the behavior of a dynamical system, but it is not always very effective for providing us with all the details of the full system. Linearization hides things like chaos and singularities, which provide interesting information about the system dynamics.

In the real world, most problems are nonlinear. In addition, most modeling problems aim to obtain information about time *and* space ($y(x_1, x_2, x_3, ..., t)$ in the general solution as opposed to only time ($y(t)$). These nonlinear functions of space and time are called *nonlinear partial differential equations* and are outside of the scope of this course. These are difficult equations, but they are important, and they are quite good at modeling many complex physical phenomena.

[1] (2,0) in this case is not actually a fixed point to the system. You will generally be given a fixed point, although this does not change anything with the solution process.