Georgia Tech

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Final list (Total= 130 points)

Math 2552 Section K

STUDENT'S NAME:

Georgia Tech ID:

Mark your Section: $K01\, or K02$, $or\ K03$, $or K04$

Writing Time: 75 min

Table of Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8.	$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
16.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$

In all the Differential Equations y means y(t), and $\frac{dy}{dt}$ means $\frac{dy(t)}{dt}$.

Question 1[20 points] Find the INVERSE LAPLACE TRANSFORM of the following function:

1 2

$$g(s) = \boxed{\frac{1}{(s^2+1)(s+2)(s+3)}} + \boxed{\frac{e^{-s}}{(s^2+4)(s+2)}}.$$

Hint: Use the Formula of Laplace Transform Convolution, see the 16th formula in the table.

$$F(s) \quad \frac{1}{(s^2+1)(s+2)(s+3)} = G(s)\,H(s)\,J(s)$$

$$\frac{1}{s^2+1} \xrightarrow{g(t)} \sin(t) \qquad \frac{1}{s+2} \xrightarrow{h(t)} e^{-2t} \qquad \frac{1}{s+3} \xrightarrow{j(t)} e^{-3t}$$

$$\left[g(t) * h(t) * j(t)\right] = 1^{st} \text{ term}$$

2nd Part

$$\frac{e^{-s}}{(s^2+4)(s+2)}$$

$$\frac{e^{-s}}{s+2} = \mathcal{L}\{u_1(t)\, e^{-2(t-1)}\}(s)$$

$$F(s) = \frac{1}{s+2} \longrightarrow e^{-2(t-1)} = f(t-c)$$

$$\underline{u_c(t)}\ \underline{f(t-c)} \longrightarrow F(s)\, e^{-cs}$$

Find this Find this

$$\frac{e^{-s}}{s+2} \longrightarrow u_1(t)e^{-2(t-1)} = n(t)$$

$$\frac{1}{s^2+4} \longrightarrow \frac{1}{2}\sin(2t) = m(t)$$

2^{nd} term: $(n * m)$

Final answer:

$$\boxed{(g * h * j) + (n * m)}$$

Question 2[20 points] Using the METHOD OF THE UNDETERMINED COEFFICIENTS find ONE particular solution of the following ODE

$$\frac{d^2y}{dt^2} - 5\frac{dy(t)}{dt} + 6y(t) = \boxed{e^{2t} + t.} \rightarrow Ate^{2t} + Bt + C = y_p$$

Hint: e^{2t} **is a solution of the HOMOGENEOUS EQUATION,** $\lambda = 2$ **is a single root..**

$y'' - 5y' + 6y = 0$

$r^2 - 5r + 6r = 0$

$(r-2)(r-3) = 0$

$r = 2, 3$

$C.S. = C_1 e^{2t} + C_2 e^{3t}$

$y_p = Ate^{2t} + Bt + C$

$y_p' = 2Ate^{2t} + Ae^{2t} + B$

$y_p'' = 2(2Ate^{2t} + Ae^{2t}) + 2Ae^{2t}$

$4Ate^{2t} + 2Ae^{2t} + 2Ae^{2t}$

$-5(2Ate^{2t} + Ae^{2t} + B)$

$+6(Ate^{2t} + Bt + C)$

$\cancel{4Ate^{2t}} + \cancel{4Ae^{2t}} - \cancel{10Ate^{2t}} - \cancel{5}Ae^{2t} - 5B$

$+\cancel{6Ate^{2t}} + \cancel{6Bt} + 6C$

$-Ae^{2t} + 6Bt + (6C - 5B) = e^{2t} + t$

$-A = 1$

$\boxed{A = -1}$

$6B = 1$

$\boxed{B = \frac{1}{6}}$

$6C - 5B = 0$

$6C = 5B$

$6C = \frac{5}{6}$

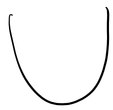
$\boxed{C = \frac{5}{36}}$

$Ate^{2t} + Bt + C$

$$\boxed{y_p(t) = -te^{2t} + \frac{1}{6}t + \frac{5}{36}}$$

Question 3[20 points] Consider the following ODE:

$y = x^2$



$$\frac{dy(t)}{dt} = y^2 - 3y + 2.$$

(a) Classify the ODE above in Liner/$\boxed{\text{Nonlinear.}}$

(b) Is the ODE above autonomous? $\boxed{\text{Yes}}$

(c) Find all the Critical/Equilibrium points of the ODE above. $\boxed{y = 1, 2}$

(d) Classify the Stability of the Equilibrium solution of this ODE.

$y^2 - 3y + 2 = (y-2)(y-1) \quad y = 1, 2$

$\boxed{\text{2 is unstable}}$

$\boxed{\text{1 is stable}}$

Question 4 [20 points] Find ONLY THE LAPLACE TRANSFORM of the solution of the following ODE:

$$\begin{cases} \frac{d^4y}{dt^4} + 2\frac{d^2y}{dt^2} + 25y = e^t\cos(t), \\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1,\ \frac{d^3y(0)}{dt^3} = 2. \end{cases}$$

I AM JUST ASKING THE LAPLACE TRANSFORM OF $y(t)$.

$$y'''' + 2y'' + 25y = e^t \cos(t)$$

$$\mathcal{L}\{y''''\} = s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)$$

$$s^4 Y(s) \quad 0 \quad + \quad 0 \quad - s \quad - 2$$

$$\mathcal{L}\{y^4\} = s^4 Y(s) - s - 2$$

$$2\mathcal{L}\{y''\} = 2(s^2 Y(s) - s y(0) - y'(0))$$

$$2(s^2 Y(s))$$

$$= 2 s^2 Y(s)$$

$$25 \mathcal{L}\{y\} = 25 Y(s)$$

$$\text{LHS: } s^4 Y(s) - s - 2 + 2s^2 Y(s) + 25 Y(s)$$

$$= Y(s)\left[s^4 + 2s^2 + 25\right] - s - 2$$

$$\text{RHS: } \mathcal{L}\{e^t \cos(t)\} = \frac{s-1}{(s-1)^2 + 1}$$

$$e^{at}\cos(bt) \Rightarrow \frac{s-a}{(s-a)^2+b^2}$$

$$Y(s)\left[s^4+2s^2+25\right]-s-2=\frac{s-1}{(s-1)^2+1}$$

$$\boxed{Y(s)=\frac{\dfrac{s-1}{(s-1)^2+1}+s+2}{s^4+2s^2+25}}$$

Question 5 [20 points] Consider the following ODE:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1}$$

(a) Find the eigenvalues of the Real matrix $2x2$ in (1).(it might be repeated.)

(b) Find the Critical Point/ equilibrium point of the ODE (1).

(c) Classify the stability of the Critical Point of the ODE (1).

(d) Find the eigenvector and one GENERALIZED EIGENVECTOR.

(e) Find all the solutions of ODE (1).

a) $\det\left|\begin{bmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix}\right| = (3-\lambda)(-1-\lambda)+4$

$-3+\lambda-3\lambda+\lambda^2+4$

$\lambda^2-2\lambda+1=0$

$(\lambda-1)(\lambda-1)=0$

$\boxed{\lambda=1,1}$

b) $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$3y_1 - 4y_2 = 0$ $\quad$ $3y_1 = 4y_2$

$\quad$ $y_1 = \frac{4}{3}y_2$

$1y_1 - 1y_2 = -1$

$\frac{4}{3}y_2 - \frac{3}{3}y_2 = -1$ $\quad$ $y_1 = -4$

$\frac{1}{3}y_2 = -1$ $\quad$ $y_2 = -3$

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}}$$

c) Classify the stability of critical pt.

eigenvalues = 1, 1 $\Longrightarrow$ $\boxed{\text{unstable node}}$

λ_1, λ_2 : Real

$\lambda_1, \lambda_2 > 0$: unstable

$\lambda_1 < 0 \quad \lambda_2 > 0$: saddle pt

$\lambda_1, \lambda_2 < 0$: stable pt.

d) $\begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

choose $y_2 = 1$

$2y_1 - 4y_2 = 0$

$2y_1 = 4y_2$

$y_1 = 2y_2$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

choose $y_2 = 0$.

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$2y_1 - 4y_2 = 2$

$y_1 - 2y_2 = 1 \qquad y_1 = 2y_2 + 1$

Eigenvector: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, Generalized Eigenvector: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

e) $\begin{bmatrix} -4 \\ -3 \end{bmatrix} + c_1 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$

Question 6 [10 points] Solve the following Initial Value:

$$\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{cases}.$$

Hint: The eigenvalues are Complex.

6) $$\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{cases}$$

$$\det\begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

$$\lambda^2 = -4$$

$$\boxed{\lambda = \pm 2i}$$

plug $2i$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2iy_1 + 2y_2 = 0$$

$$2y_2 = 2iy_1 \qquad \begin{bmatrix} 2 \\ 2i \end{bmatrix}$$

$$y_2 = iy_1 \qquad \text{choose } y_1 = 2$$

eigenvectors: $\begin{bmatrix} 2 \\ \pm 2i \end{bmatrix}$

$$\begin{cases} R & e^{0t}\begin{bmatrix} 2\cos(2t) \\ -2\sin(2t) \end{bmatrix} + \\ I & e^{0t}\begin{bmatrix} 2i\sin(2t) \\ 2i\cos(2t) \end{bmatrix} \end{cases}$$

$$e^{0t}\left(\cos(2t) + i\sin(2t)\right)\begin{bmatrix} 2 \\ 2i \end{bmatrix}$$

$$C_1\begin{bmatrix} 2\cos(2t) \\ -2\sin(2t) \end{bmatrix} + C_2\begin{bmatrix} 2\sin(2t) \\ 2\cos(2t) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$2C_1\cos(2t) + 2C_2\sin(2t) = 0$$
$$-2C_1\sin(2t) + 2C_2\cos(2t) = 2$$

$t = 0$

$$2C_1 = 0$$
$$2C_2 = 2$$

$\boxed{C_2 = 1}$ $\boxed{C_1 = 0}$

$$1\begin{bmatrix} 2\sin(2t) \\ 2\cos(2t) \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

Question 7 [10 points] Consider the following Nonlinear ODE system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2x_1(t) + x_2(t) \\ 2x_2(t) + x_1(t)^2 \end{bmatrix} \qquad (2)$$

$0 = 2x_1 + x_2$ $\quad x_2 = -2x_1$ $\quad x_2 = -2(0) = 0 \to (0,0)$

$0 = 2x_2 + x_1^2$

$2(-2x_1) + x_1^2 = 0$

$-4x_1 + x^2 = 0$ $\quad x_1(-4 + x_1) = 0$ $\quad x_1 = 0,\ x_1 = 4 \to (4,-8)$ $\quad x_2 = -2(4) = -8$

(a) Find all the critical points.

$\boxed{(0,0)}$ $\boxed{(4,-8)}$

(b) Find the almost linear ODE representation for (2) in the neighborhood of the critical point $(x_1, x_2) = (0,0)$ **I only want the LINEAR PART, you can denote the remainder function by** $g(t, x_1, x_2)$.

(c) Classify the Stability for all the Critical Points.

b) Jacobian $F = \sim\sim\sim$

$G = \sim\sim\sim$

$$\begin{bmatrix} F_{x_1} & F_{x_2} \\ G_{x_1} & G_{x_2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2x_1 & 2 \end{bmatrix}$$

$$(0,0) \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g(t, x_1, x_2)}$$

c) $(0,0) \to \lambda = 2, 2 \to$ unstable

$$(4,-8) \to \begin{bmatrix} 2 & 1 \\ 8 & 2 \end{bmatrix} \to (2-\lambda)(2-\lambda) - 8$$

$$\lambda^2 - 4\lambda - 4 = 0$$

$$\lambda = \frac{4 \pm \sqrt{32}}{2}$$

$>0,\ <0$

$\boxed{\text{Unstable saddle pt}}$

Question 8 [10 points] Consider the following Initial Value Problem:

$$\begin{cases} \frac{dy(t)}{dt} = 4y(t)^2 + t, \\ y(0) = 1. \end{cases} \tag{3}$$

For the STEP SIZE $\delta = 0.5$, find the value of the approximate solution y_{ap} of (3) on $t = 1.5$ using THE EULER's METHOD.

$y_1(t) = y(0) + \Delta f(0, y(0))$

$= 1 + .5(4y^2 + t) = 1 + .5(4) = 3$

$y_2 = 3 + .5(4y^2 + t) = 3 + .5(4(3)^2 + .5)$

$= 3 + .5(36.5) = 3 + 18.25 = 21.25$

$y_3 = 21.25 + .5(4y^2 + t) =$

$21.25 + .5(4(21.25)^2 + 1)$

$\boxed{= 924.875}$