Tomorrow Midterm @ 2 PM

Office Hours 12 - 2 PM

Math Lab Clough 280

Howey physics L2

Canvas Midterm Review

Extra Review @

6PM Skiles 170

---

1) a) $\frac{dy}{dt} = e^{-2y}$

$$e^{2y} \, dy = 1 \, dt$$

int. both sides

$$\boxed{\frac{1}{2} e^{2y} = t + C}$$

b) Let $f(y) = y^2 + y - 2$

i) Find eq. points

$$0 = y^2 + y - 2$$

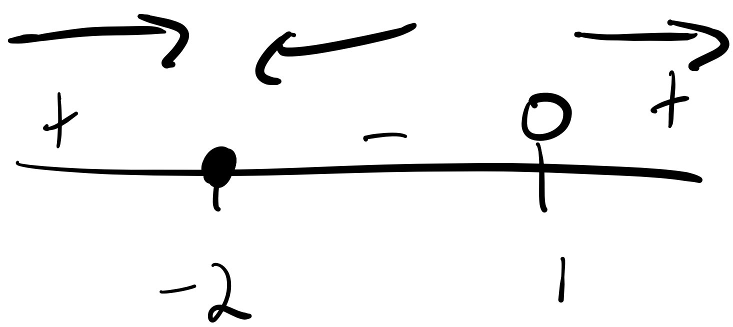
$$0 = (y+2)(y-1)$$

$$\boxed{y = -2,\ 1}$$

$$f(y) = \frac{dy}{dt} = y^2 + y - 2$$

ii)

iii)



$$\boxed{\begin{array}{lll} -2 & \text{is} & \text{stable} \\ 1 & \text{is} & \text{unstable} \end{array}}$$

Q2) $x_1' = 3x_1(t) - 2x_2(t)$

$x_2' = 4x_1(t) - x_2(t)$

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}$$

$(3-\lambda)(-1-\lambda)+8$

$-3+\lambda-3\lambda+\lambda^2+8$

$\lambda^2-2\lambda+5$

$$\lambda=\frac{2\pm\sqrt{4-4(1)(5)}}{2(1)}$$

$$\lambda=\frac{2\pm\sqrt{4-20}}{2}$$

$$\lambda=\frac{2\pm\sqrt{-16}}{2}$$

$$\lambda=\frac{2\pm 4i}{2}=\underline{\underline{1\pm 2i}}$$

$1+2i$

$$\begin{bmatrix} 3-1-2i & -2 \\ 4 & -1-1-2i \end{bmatrix}$$

$$= \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(2-2i)\eta_1 - 2\eta_2 = 0$$

$$2\eta_2 = (2-2i)\eta_1$$

$$\eta_2 = (1-i)\eta_1$$

$$v_1 = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ (1-i)\eta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ (1-i) \end{bmatrix}$$

choose $\eta_1 = 1$

$$= e^{(1+2i)t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$e^{t}\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$\mathbb{R} \quad C_1 e^{t} \begin{bmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} +$$

$$+ \quad \mathbb{I} \quad C_2 e^{t} \begin{bmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{bmatrix} \quad \text{ok}$$

$$\text{ok} \quad C_1 e^{t}\left(\cos(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$$+ C_2 e^{t}\left(\sin(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

---

3) $\begin{cases} \dfrac{dy(t)}{dt} + \dfrac{1}{(t+1)(t+2)} y = t \\ y(0) = 2 \end{cases}$

$\rightarrow y' + P(t) y = H(t)$

$e^{\int P(t)\,dt}$

$e^{\int \frac{1}{(t+1)(t+2)}\,dt}$

PFD

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$$1 = A(t+2) + B(t+1)$$

$$1 = At + 2A + Bt + B$$

$$1 = (A+B)t + (2A+B)$$

$$A + B = 0$$

$$2A + B = 1$$

$$B = 1 - 2A$$

$A + 1 - 2A = 0$

$-A + 1 = 0$

$\boxed{A = 1}$

$\boxed{B = -1}$

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$$= \frac{1}{t+1} - \frac{1}{t+2}$$

I.F.: $\int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt$

e

$$= e^{\ln(|t+1|) - \ln(|t+2|)}$$

$$\cancel{e^{\ln}}\left(\left|\frac{t+1}{t+2}\right|\right)$$

$$\text{I.F.} = \frac{t+1}{t+2}$$

$$\frac{dy(t)}{dt} + \frac{1}{(t+1)(t+2)}\,y = t$$

$$\frac{t+1}{t+2}\,y' + \frac{t+1}{t+2}\,\frac{1}{(t+1)(t+2)}\,y = t\left(\frac{t+1}{t+2}\right)$$

$t+2 \qquad t+2 \quad (t+1)(t+2)$

LHS $= [y\,\mu(t)]'$

$$\left(y\,\frac{t+1}{t+2}\right)' = t\left(\frac{t+1}{t+2}\right).$$

$$y\,\frac{t+1}{t+2}$$

$$= \frac{t^2+t}{t+2}$$

$$t - 1 + \frac{2}{t+2}$$

$$t+2 \,\overline{)\,t^2 + t}$$

$$-\ t^2 + 2t$$

$$-t$$

$$-\ -t - 2$$

$$2$$

$$\int \left(t - 1 + \frac{2}{t+2}\right) dt$$

$$y\,\frac{t+1}{t+2} = \frac{1}{2}t^2 - t + 2\ln(|t+2|) + C$$

$$y = \left(\frac{t+2}{t+1}\right)\left(\frac{1}{2}t^2 - t + 2\ln(|t+2|) + C\right)$$

$$y(0) = 2$$

$$2 = 2\left(0 - 0 + 2\ln(|2|) + C\right)$$

$$1 = 2\ln(|t+2|) + C$$

$$C = 1 - 2\ln(|t+2|)$$

$$C = 1 - 2\ln(|2|)$$

$$\boxed{y = \left(\frac{t+2}{t+1}\right)\left(\frac{1}{2}t^2 - t + 2\ln(|t+2|) + 1 - 2\ln(2)\right)}$$

Q4) $y'' - y' + y = 2$

1) 2 1^{st} order ODE's

2) Find solutions

1) $\left.\begin{array}{l} x_1 = y \\ x_2 = y' \end{array}\right\} \begin{array}{l} x_1' = ? \\ x_2' = ? \end{array}$

$$x_1' = y' = x_2$$

$$x_2 = y = y - y + 2$$

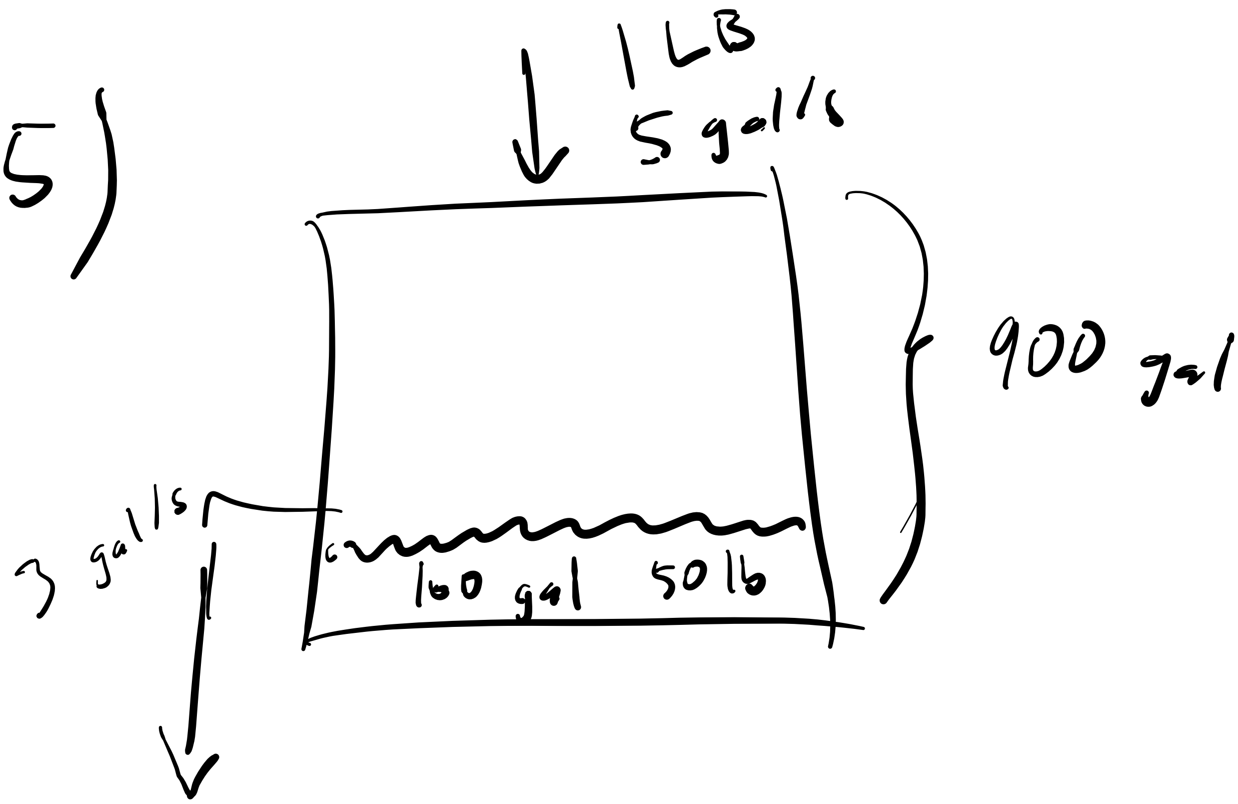
$$= x_2 - x_1 + 2$$

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_2 - x_1 + 2 \end{aligned}}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) shown on page 19

5)



$$\frac{dQ}{dt} = in - out$$

$$= 1(5) - 3\left(\frac{Q(t)}{100+5t-3t}\right)$$

$$\frac{dQ}{dt} = 5 - 3\frac{Q(t)}{100+2t}$$

$$\frac{dQ}{dt} + \frac{3Q(t)}{100+2t} = 5$$

$$\int\left(\frac{3}{100+2t}\right)dt$$

e

$$\int\frac{3}{100+2t}\,dt \qquad u = 100+2t$$

$$du = 2\,dt \qquad dt = \frac{du}{2}$$

$$\frac{3}{2}\int \frac{1}{u}\,du = \frac{3}{2}\ln|100+2t|$$

I.F.

$$e^{\frac{3}{2}\ln|100+2t|}$$

$$= \cancel{e^{\ln}}(100+2t)^{3/2}$$

$$= (100+2t)^{3/2}$$

$$(100+2t)^{3/2}Q' + (100+2t)^{3/2}\left(\frac{3}{100+2t}\right)Q$$

$$= 5(100+2t)^{3/2}$$

$[\mu(x)\,Q]'$ is LHS

$$\int\left[(100+2t)^{3/2}Q\right]' = 5(100+2t)^{3/2} + C$$

$$(100+2t)^{3/2} Q = (100+2t)^{5/2} + C$$

$$Q(0) = 50$$

$$(100)^{3/2} Q = (100)^{5/2} + C$$

$$50{,}000 = 100{,}000 + C$$

$$C = -50{,}000$$

$$(100+2t)^{3/2} Q = (100+2t)^{5/2} - 50{,}000$$

t ?

$100 \to 900$

$$\boxed{\frac{800\ \text{g}}{(5-3)\ \text{g/s}}} = \text{s}$$

$$\frac{g}{g/s} = \cancel{\frac{g}{1} \cdot \frac{s}{g}} = s\ ?$$

$$\frac{800}{2} = \underline{400}\ s$$

$$(100+2t)^{3/2} Q = (100+2t)^{5/2} - 50{,}000$$

$t = 400$: Plug $t = 400$ into this eqn:

$$\boxed{898.148 \text{ lbs salt}}$$

A x + 6

4b) $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$0x_1 + x_2 = 0$ $\qquad$ $x_2 = 0$

$-x_1 + x_2 = -2$ $\qquad$ $x_1 = 2$

Eq. pt. $(2, 0)$

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$\|$

$$\Downarrow$$

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}$$

$$= (0-\lambda)(1-\lambda) + 1$$

$$-\lambda + \lambda^2 + 1$$

$$\lambda^2 - \lambda + 1$$

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$$

Choose $\frac{1 + \sqrt{3}i}{2}$ eigenvalue

Find eigenvector:

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1-\sqrt{3}i}{2} & 1 \\ -1 & 1 - \frac{1-\sqrt{3}i}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{-1-\sqrt{3}i}{2} & 1 \\ -1 & 1-\frac{1-3i}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{-1-\sqrt{3}i}{2}\right)\eta_1 + \eta_2 = 0$$

$$\eta_2 = \frac{1+\sqrt{3}i}{2}\eta_1$$

choose $\eta_1 = 1$

$\frac{1+\sqrt{3}i}{2}$

$\frac{1}{2} + \frac{\sqrt{3}i}{2}$

$$\begin{bmatrix} \eta_1 \\ \left(\frac{1+\sqrt{3}i}{2}\right)\eta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{3}i}{2} \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + e^{\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + i\sin\left(\frac{\sqrt{3}}{2}t\right)\right)\begin{bmatrix} 1 \\ \frac{1+\sqrt{3}i}{2} \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} +$$

$$C_1 e^{t/2} \begin{bmatrix} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{bmatrix} +$$

$$C_2 e^{t/2} \begin{bmatrix} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) \end{bmatrix}$$

6)

$$x_1' = x_1 - 2x_2$$

$$x_2' = -2x_1 + x_2$$

1) $$\begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}$$

$$= \lambda^2 - 2\lambda - 3$$

$$(\lambda - 3)(\lambda + 1)$$

$$\boxed{\lambda = 3, -1}$$

2) 3 is unstable

-1 is stable

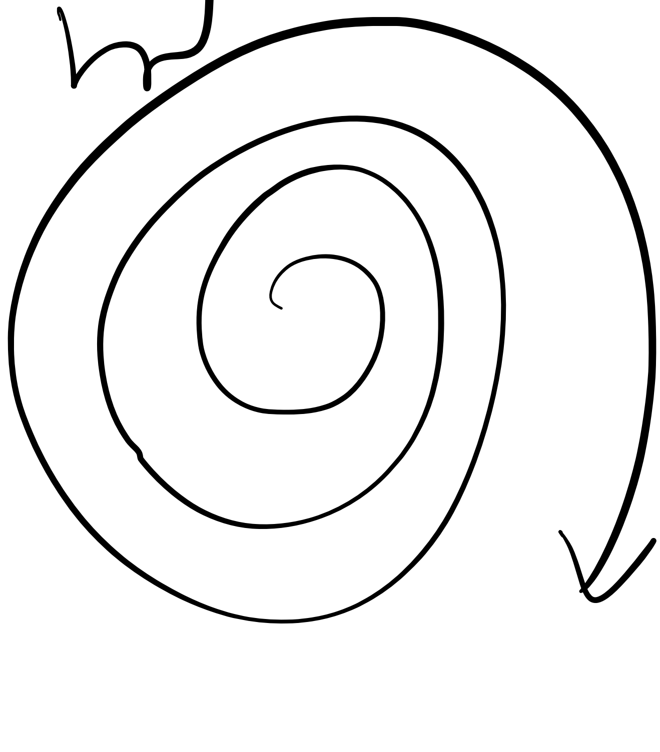
System is an unstable saddle pt.

Extra Review!

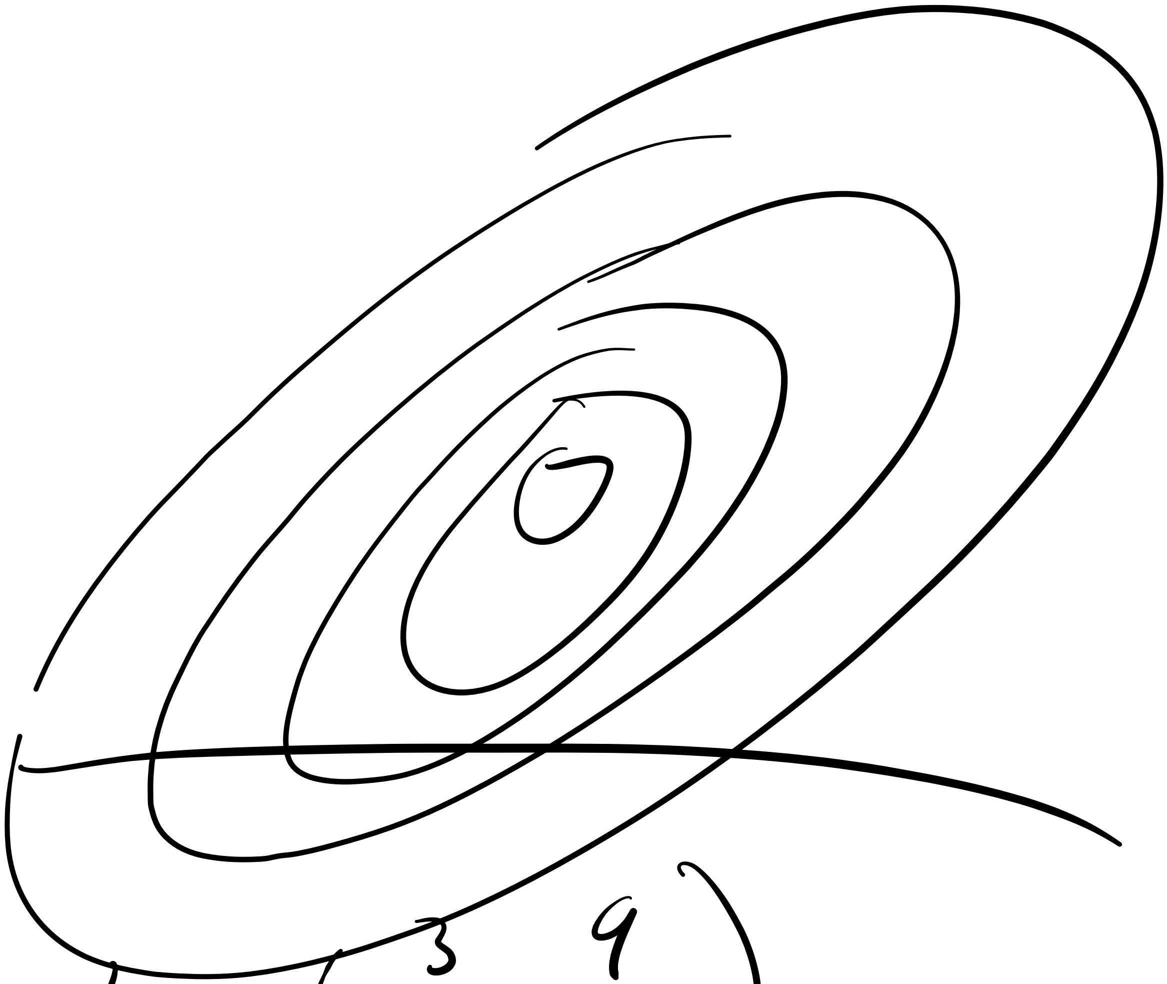
Complex Phase Portrait

$\lambda = -2 \pm 3i$

$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$



$\lambda = 5i$



$$x' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}$$

$$y(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{pmatrix}$$

$(3-\lambda)(-3-\lambda)+36$

$-9+\cancel{3\lambda}\cancel{-3\lambda}+\lambda^2+36$

λ^2+27

$= \pm 3\sqrt{3}\,i$

$0 \pm 3\sqrt{3}\,i$

$a \pm b\,i$

0

circle,
ellipse

$3\sqrt{3}\,i$

$$X' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 3-3\sqrt{3}i & 9 \\ -4 & -3-3\sqrt{3}i \end{pmatrix}$$

$$\begin{pmatrix} 3-3\sqrt{3}i & 9 \\ -4 & -3-3\sqrt{3}i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(3-3\sqrt{3}i)\eta_1 + 9\eta_2 = 0$$

$$9\eta_2 = -(3-3\sqrt{3}i)$$

$$\eta_2 = -\frac{1}{3}(1-\sqrt{3}i)\eta_1$$

$$\eta_1$$

$$v_1 = \begin{pmatrix} -\frac{1}{3}(1-\sqrt{3}i)\eta_1 \end{pmatrix}$$

choose

$\eta_1 = 3$

$$v_1 = \begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix}$$

$$e^{\lambda_1 t} v_1$$

$$e^{(3\sqrt{3}i)t} \begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix}$$

$a+bi$

$0 + 3\sqrt{3}i$

$\cancel{\left(e^{at}\right)}\left(\cos(bt) + i\sin(bt)\right)(V_1)$

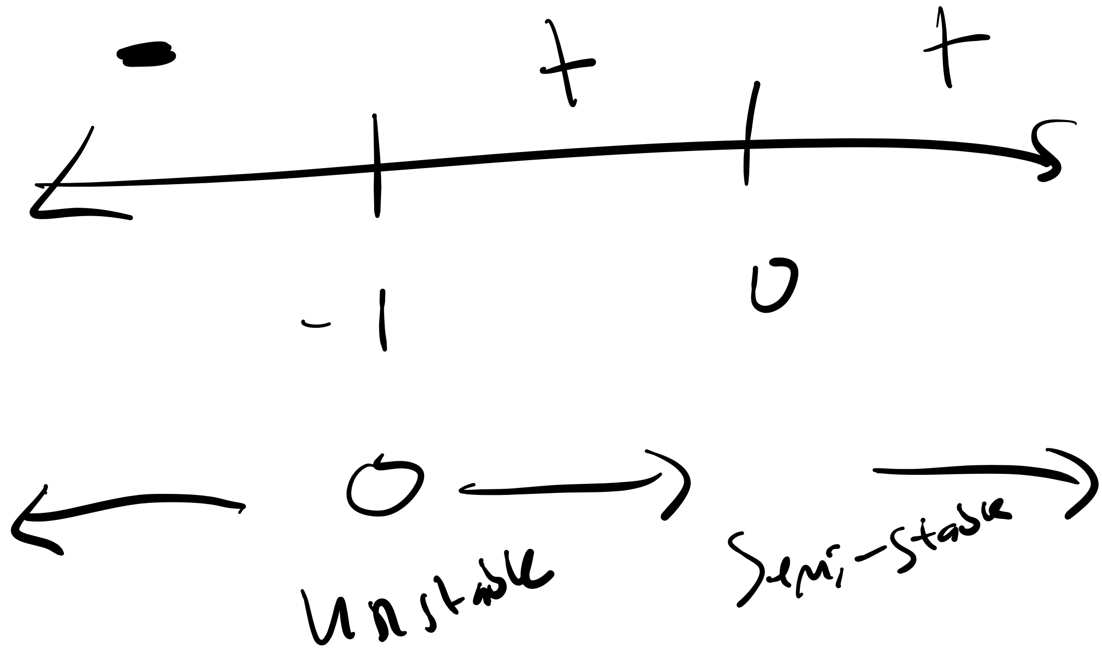
$\cos(3\sqrt{3}t) + i\sin(3\sqrt{3}t)\begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix}$

Expand out

1) $y^2(y+1)$.

1) Eq. points

$\boxed{y = 0, -1}$



$$\frac{dy}{dt} + \frac{1}{t+2} y = t^2$$

$$\int \frac{1}{t+2} \, dt$$

$$e$$

$$\int \frac{1}{t+2} = \ln(|t+2|)$$

$$\cancel{e^{\ln(t+2)}}$$

I.F. $\quad t+2$

$$(t+2)\frac{dy}{dt} + \frac{(t+2)}{t+2}\,y = t^2\,(t+2)$$

$$\int [y(t+2)]' = \int t^2(t+2)$$

$$y(t+2) = t^3 + 2t^2 + C$$

$$y(t+2) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + C$$

$$y = \frac{\frac{1}{4}t^4 + \frac{2}{3}t^3 + C}{t+2}$$

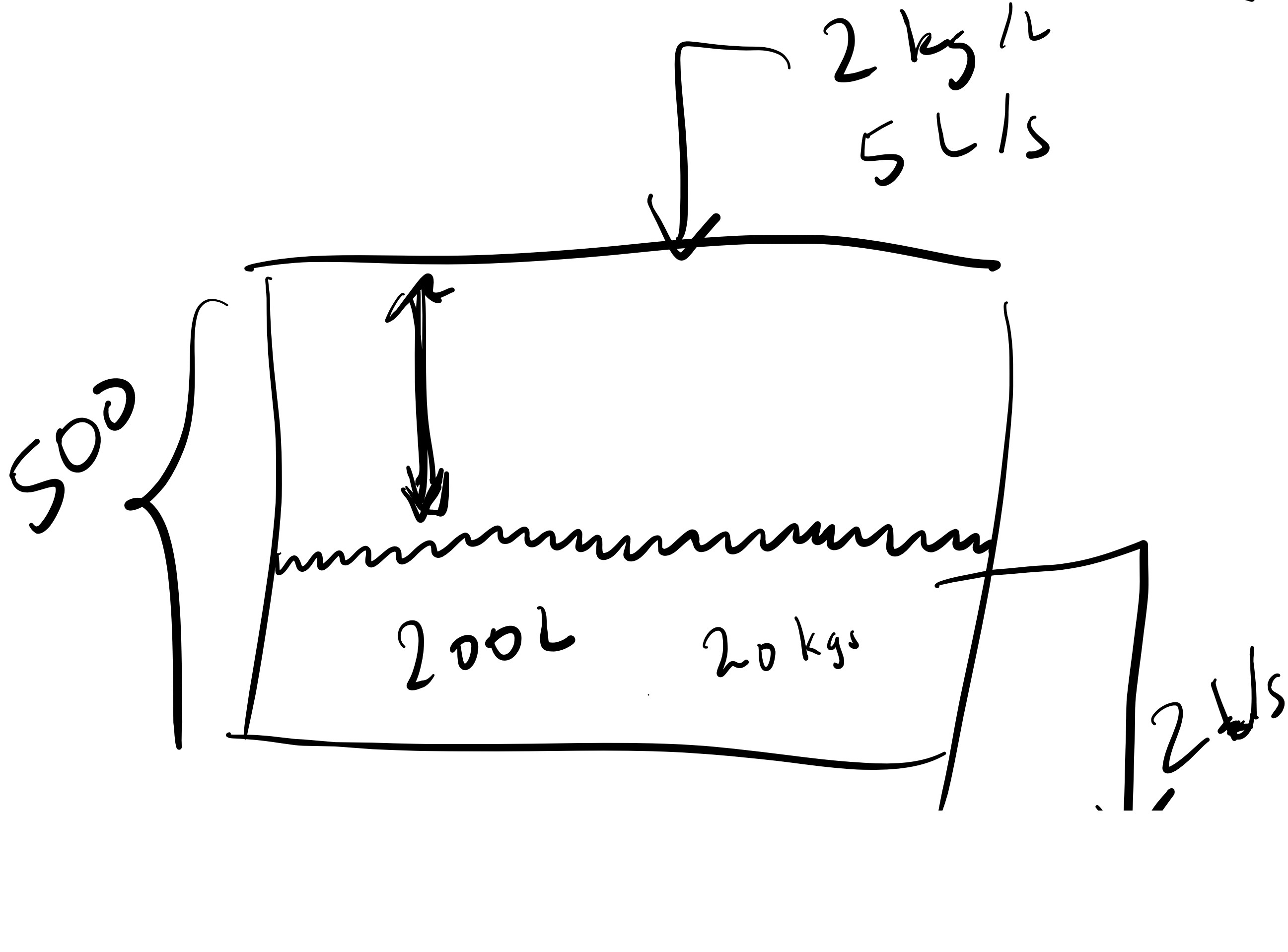
$$y(0) = 7$$

$$7 = \frac{0 + 0 + C}{2}$$

$$7 = \frac{C}{2}$$

$$\boxed{C = 14}$$

$$y = \frac{\frac{1}{4}t^4 + \frac{2}{3}t^3 + 14}{t+2}$$



i) $\frac{dQ}{dt} = 2(5) - 2\left(\frac{Q(t)}{200+5t-2t}\right)$

$$\boxed{\frac{dQ}{dt} = 10 - \frac{2Q(t)}{200+3t}, \quad Q(0) = 20}$$

ii) $\frac{300 \text{ L}}{3 \text{ L/s}} = ?\text{ s}$

$\frac{300 \text{ L}}{3 \text{ L/s}} = \boxed{100 \text{ s}}$

$$\frac{dQ}{dt} = 10 - \frac{2Q(t)}{200+3t},$$

$$Q(0) = 20$$

$$\frac{dQ}{dt} + \frac{2}{200+3t} Q = 10$$

$$\int \frac{2}{200+3t} \, dt$$

$$e$$

$$2 \int \frac{1}{200+3t} =$$

$$+3t))$$

$$\frac{2}{3}\ln(|200\ldots$$

$$e$$

$$\cancel{e^{\ln(200+3t)^{2/3}}}$$

$$I.F. = (200+3t)^{2/3}$$

$$y(200+3t)^{2/3} = \int 10(200+3t)^{2/3}$$

$$y(200+3t)^{2/3} = 2(200+3t)^{5/3} + C$$

$$y(0)=20$$

$$\ldots^{2/3} \ldots)^{5/3}$$

$$20(200) = 2(200) + C$$

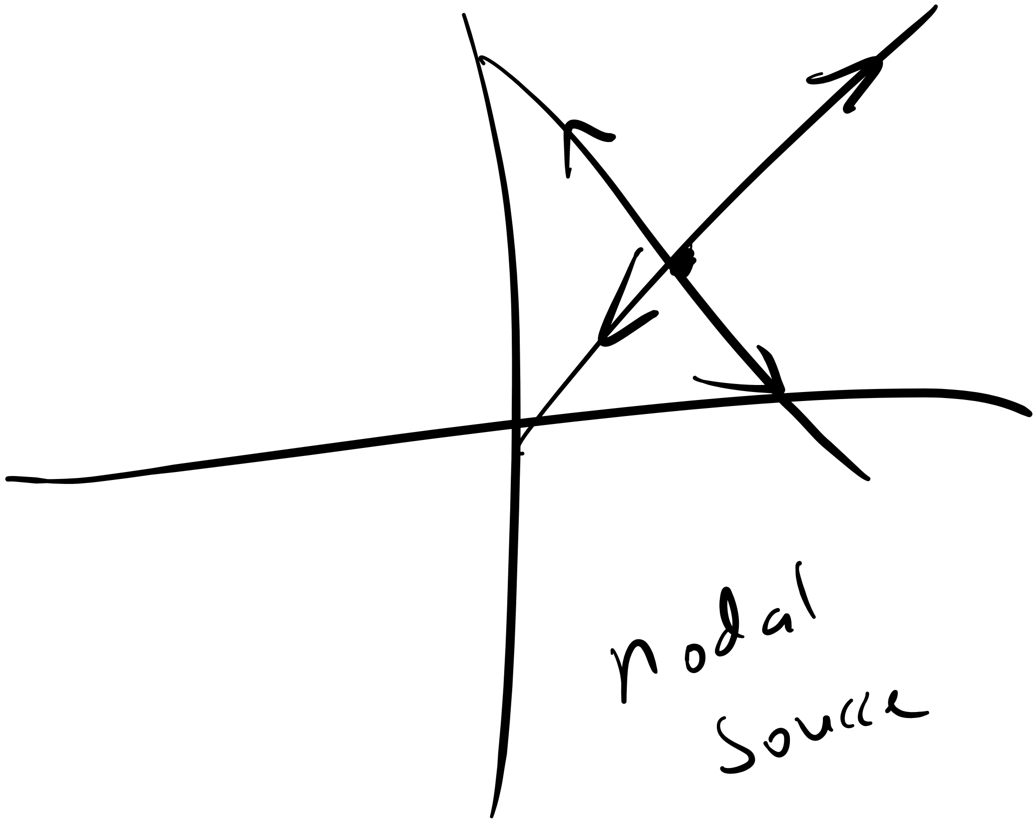
IVP Solution: $C = -13{,}000$

$$y(200+3t)^{2/3} = 2(200+3t)^{5/3} - 13{,}000$$

$$y(200 + 3(100))^{2/3} =$$

$$2(200+3(100))^{5/3}$$

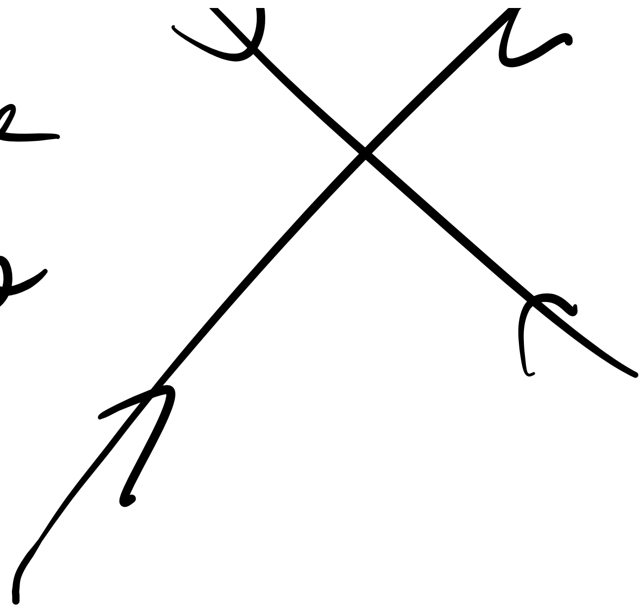
$$-13{,}000$$



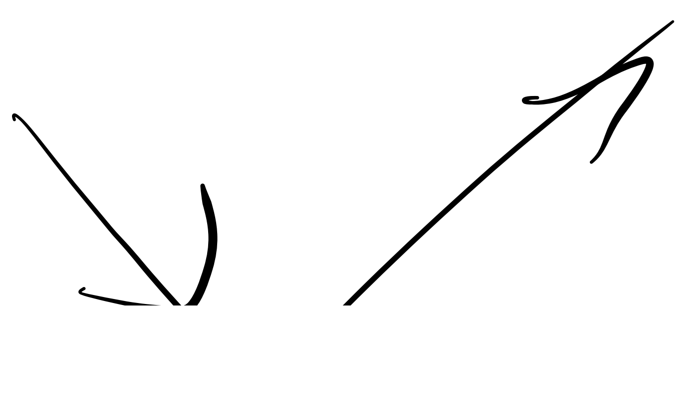
nodal
source

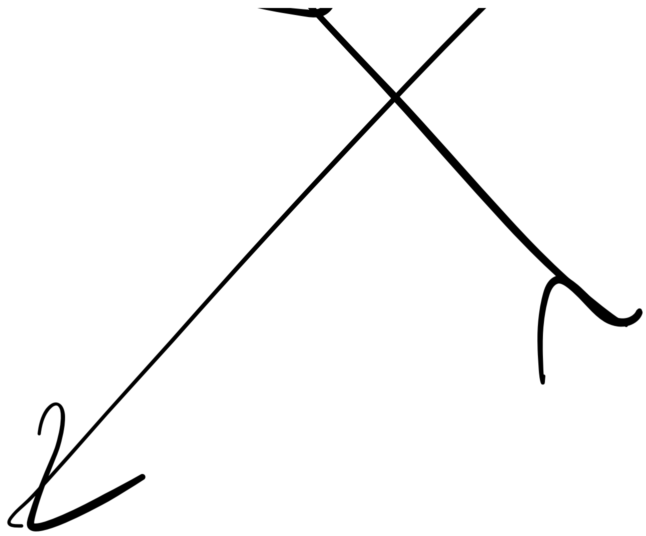
unstable
node

Stable node



nodal sink





Saddle point

unstable

PFD

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$1 = A(x+3) + B(x+2)$$

set $x = -3$

$$1 = A(0) + B(-1)$$

$$1 = -B$$

$$\boxed{B = -1}$$

$$1 = A(1)$$

$$\boxed{A = 1}$$

$$\frac{1}{x+2} - \frac{1}{x+3}$$

Repeated Eigenvalue:

$$x' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} x$$

$$x(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Find λ:

$$\begin{pmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix}$$

$$(7-\lambda)(3-\lambda) + 4$$

$$21 - 3\lambda - 7\lambda + \lambda^2 + 4$$

$$\lambda^2 - 10\lambda + 25$$

$$(\lambda - 5)(\lambda - 5)$$

$$\lambda_{1,2} = 5$$

$$\begin{pmatrix} 7-5 & 1 \\ -4 & 3-5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2\eta_1 + 1\eta_2 = 0$$

$$2\eta_1 = -\eta_2$$

$$\eta_1 = -\frac{1}{2}\eta_2$$

$$\begin{pmatrix} -\frac{1}{2}\eta_2 \\ \eta_2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\eta_2 = -2$$

First eigenvector

12

G.E.:

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$2\eta_1 + 1\eta_2 = 1$$

$$\eta_2 = 1 - 2\eta_1$$

$$\begin{pmatrix} \eta_1 \\ 1-2\eta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Choose $\eta_1 = 0$.

G.S.

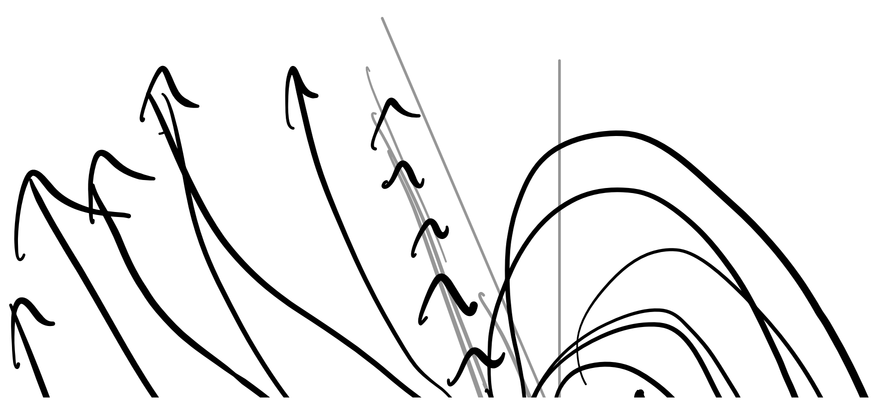
$$\boxed{y = C_1 e^{5t}\begin{pmatrix}1\\-2\end{pmatrix} + C_2 e^{5t}\left(t\begin{pmatrix}1\\-2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right)}$$

$y(0) = 2$

$$2 = C_1\begin{pmatrix}1\\-2\end{pmatrix} + C_2\begin{pmatrix}0\\1\end{pmatrix}$$

$2 = 1C_1 + 0$ $\quad \boxed{C_1 = 2}$

$2 = -2C_1 + 1C_2$ $\quad \boxed{C_2 = 6}$



$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

improper
unstable

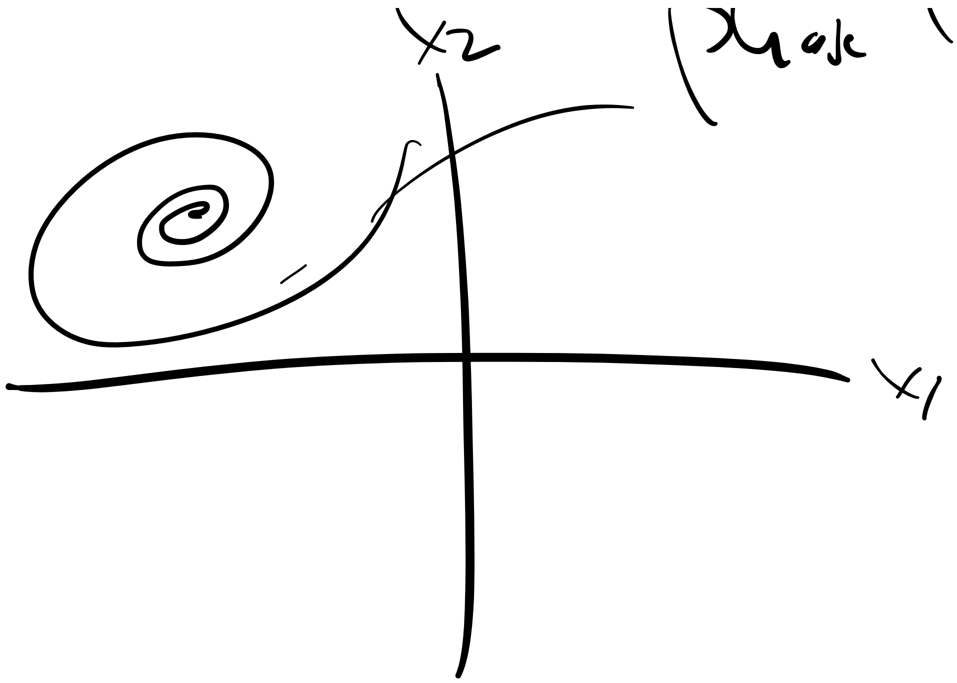
$$\left(c_1 e^{2t} u_1 + c_2 e^{3t} u_2\right)$$

5) Classify the stability of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$y^2(y+1)=0$$

Phase Line

Phase Portrait



Direction Fields