April 1, 2024

Midterm 2 Review

Q1) find the inverse Laplace Transform

$$g(s) = \frac{1}{(s^2+1)(s+2)(s+3)}$$

PFD:

$$\frac{1}{(s^2+1)(s+2)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{s+3}$$

$$1 = (As+B)(s+2)(s+3)$$
$$+ C(s^2+1)(s+3)$$
$$+ D(s^2+1)(s+2)$$

$A = -\frac{1}{10}$, $B = \frac{1}{10}$, $C = \frac{1}{5}$, $D = -\frac{1}{10}$

$$\frac{1}{(s^2+1)(s+2)(s+3)} =$$

$$-\frac{s}{10(s^2+1)} + \frac{1}{10(s^2+1)} + \frac{1}{5(s+2)} - \frac{1}{10(s+3)}$$

$\mathcal{L}^{-1} \Rightarrow$

$$y_p(t) = \boxed{-\frac{1}{10}\cos(t) + \frac{1}{10}\sin(t) + \frac{1}{5}e^{-2t} - \frac{1}{10}e^{-3t}}$$

Q2 U.C.

$$y'' - 10y' + 25y = e^{5t} + t^2 + 1$$

C.S. $r^2 - 10r + 25 = 0$

$(r-5)(r-5)$

$r = 5, 5$

$= C_1 e^{5t} + C_2 t e^{5t}$

$$At^2 e^{5t} + (Bt^2 + Ct + D)$$

$$y_{p(t)} = At^2 e^{5t} + Bt^2 + Ct + D$$

$$y'_{p(t)} = A(5t^2 e^{5t} + 2te^{5t}) + 2Bt + C$$

$$= 5At^2e^{5t} + 2Ate^{5t} + 2Bt + C$$

$$y_{p(1)}'' = A\left(25t^2e^{5t} + 10te^{5t} + 10te^{5t} + 2e^{5t}\right)$$
$$+ 2B$$

$$= 25At^2e^{5t} + 10Ate^{5t} + 10Ate^{5t}$$
$$+ 2Ae^{5t} + 2B$$

$$y'' - 10y' + 25y = e^{5t} + t^2 + 1$$

$$\cancel{25At^2e^{5t}} + \cancel{10Ate^{5t}} + \cancel{10Ate^{5t}}$$
$$+ 2Ae^{5t} + 2B$$
$$- \cancel{50At^2e^{5t}} - \cancel{20Ate^{5t}}$$

$$-20Bt - 10C$$

$$+ \cancel{25At^2 e^{5t}} + 25Bt^2$$

$$+ 25Ct + 25D$$

$$= 2Ae^{5t} + 25Bt^2$$

$$+ (-20B + 25C)t$$

$$+ (2B - 10C + 25D)$$

$$= e^{5t} + t^2 + 1$$

$$2A = 1$$

$$25B = 1$$

$$-20B + 25C = 0$$

$$2B - 10C + 25D = 1$$

$$\boxed{A = \frac{1}{2} \quad B = \frac{1}{25} \quad C = \frac{4}{125} \quad D = \frac{31}{625}}$$

$y_p =$

$$\boxed{\frac{1}{2}t^2e^{5t} + \frac{1}{25}t^2 + \frac{4}{125}t + \frac{31}{625}}$$

Q3)

$$y'' + 4y' + 4y = e^{-2t}t$$

a) $r^2 + 4r + 4 = 0$

$$(r+2)(r+2) = 0$$

$$r = -2, -2$$

$$\boxed{C.S. = y_c = C_1 e^{-2t} + C_2 t e^{-2t}}$$

b) $\begin{bmatrix} y_p(t) \\ y_p'(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix}$

$$y_p = -y_1 \int \frac{y_2 g(t)}{W} + y_2 \int \frac{y_1 g(t)}{W}$$

$$\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix}$$

$$W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$= e^{-2t}\left(e^{-2t} - 2te^{-2t}\right)$$

$$- te^{-2t}\left(-2e^{-2t}\right)$$

$$= e^{-4t} \cancel{- 2te^{-4t}}$$

$$\cancel{+ 2te^{-4t}}$$

$$= e^{-4t} = W$$

$$g(t) = e^{-2t} t$$

$$y_1 = e^{-2t} \qquad y_2 = te^{-2t}$$

$$te^{-2t} te^{-2t} = t^2 e^{-4t}$$

$$y_p = -y_1 \int \frac{y_2 g(t)}{W} + y_2 \int \frac{y_1 g(t)}{W}$$

$$= -e^{-2t} \int t^2 \, dt + te^{-2t} \int t \, dt$$

$$= -e^{-2t} \left[\frac{1}{3} t^3 \right] + te^{-2t} \left[\frac{1}{2} t^2 \right]$$

$$\boxed{y_p = -\frac{1}{3}e^{-2t}t^3 + \frac{1}{2}t^3e^{-2t}}$$

Q4)

Laplace:

$$y'' + 10y' + 25y = e^t\cos(t),$$

$$y(0) = 2 \qquad y'(0) = 1.$$

$$s^2Y(s) - s2 - 1$$
$$+10(sY(s) - 2)$$
$$+25Y(s).$$

$\mathcal{L}\{e^t\cos(t)\}$

$\Rightarrow \mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$

$$\frac{s-1}{(s-1)^2+1}$$

$$s^2Y(s) - 2s - 1 + 10sY(s) - 20 + 25Y(s) = \frac{s-1}{(s-1)^2+1}$$

$$Y(s)\left[s^2+10s+25\right]-2s-21=\frac{s-1}{(s-1)^2+1}$$

$$\boxed{Y(s)=\frac{\frac{s-1}{(s-1)^2+1}+2s+21}{s^2+10s+25}}$$

5) $g(t) = \begin{cases} 3 & \text{if } 0 \le t \le 1 \\ 1 & \text{if } 1 < t \le 2 \\ 4 & \text{if } 2 < t < \infty \end{cases}$

$\Downarrow$

$3 \longrightarrow \text{shift} \longrightarrow \text{shift}$

$3 + (1-3)u_1 + (4-1)u_2$

$3 - 2u_1(t) + 3u_2(t)$

$\Downarrow \mathcal{L}$

$$\boxed{\frac{3}{s} - 2\frac{e^{-s}}{s} + 3\frac{e^{-2s}}{s}}$$

6)

$$x^2 y'' + 3xy' + 4y = 0$$

$$y(1) = 1 \qquad y'(1) = 1$$

$$\downarrow$$

$$\underset{A}{y''} + \underset{(B-A)}{2y'} + \underset{C}{4y} = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= -1 \pm \sqrt{-3}$$

$$-1 \pm \sqrt{3}i$$

$$y(t) = e^{-t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right)$$

$$y(1) = 1 \qquad e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$

$$y(x) = \frac{1}{x}\left(C_1 \cos(\sqrt{3}\ln(x)) + C_2 \sin(\sqrt{3}\ln(x))\right)$$

$$y(1) = 1$$

$$1 = 1\left(C_1 \cos(0) + C_2 \sin(0)\right)$$

$$\boxed{1 = C_1}$$

$$y(x) = \frac{1}{x}\left(C_1 \cos(\sqrt{3}\ln(x)) + C_2 \sin(\sqrt{3}\ln(x))\right)$$

$$y(t) = e^{-t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right)$$

$$y(t) = C_1 e^{-t}\cos(\sqrt{3}t) + C_2 e^{-t}\sin(\sqrt{3}t)$$

$$y'(t) = C_1\left(-\sqrt{3}e^{-t}\sin(\sqrt{3}t) - e^{-t}\cos(\sqrt{3}t)\right)$$
$$+ C_2\left(\sqrt{3}e^{-t}\cos(\sqrt{3}t) - e^{-t}\sin(\sqrt{3}t)\right)$$

$$t = \ln(x)$$

$$y'(x) = C_1\left(-\sqrt{3}\left(\frac{1}{x}\right)\sin(\sqrt{3}\ln(x))\right.$$
$$\left. - \frac{1}{x}\cos(\sqrt{3}\ln(x))\right)$$
$$+ C_2\left(\sqrt{3}\,\frac{1}{x}\cos(\sqrt{3}\ln(x))\right.$$
$$\left. - \frac{1}{x}\sin(\sqrt{3}\ln(x))\right)$$

$y'(1) = 1$

$$1 = C_1\left(-\frac{1}{x}\right) + C_2\left(\frac{\sqrt{3}}{x}\right)$$

$$x \to 1$$

$$1 = -C_1 + \sqrt{3}\,C_2$$

$$1 = -1 + \sqrt{3}\,C_2$$

$$2 = \sqrt{3}\,C_2$$

$$C_2 = \frac{2}{\sqrt{3}}$$

$$\boxed{y(x) = \frac{1}{x}\left(1\cos(\sqrt{3}\ln(x)) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}\ln(x))\right)}$$

Extra

$$y'' + \sim y' + \sim = \boxed{\sim}$$

$\lambda = 2, 2$

C.S. $\Rightarrow ae^{2t} + ate^{2t}$

RHS guess

$\cancel{ae^{2t}}, \cancel{ate^{2t}}, at^2e^{2t}$

$$y'' + \sim y' + \sim y = \sim$$

Variation of parameters

$\underbrace{y_1, \ y_2,}$ W, $g(t)$

$y'' + \xi y' + \xi y = 0$

$r = \lambda_1 , \lambda_2$

Gs. $\longrightarrow$ $y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$y_1 \qquad y_2$

$$W = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

VoP:

$$-y_1 \int \frac{y_2 \, g(t)}{W} + y_2 \int \frac{y_1 \, g(t)}{W} = y_p$$

Ex.

Find the Laplace transform:

$$x(t) = \begin{cases} 5 & 0 < t \le 2 \\ 4 & 2 \le t \le 6 \\ 2 & 6 \le t \le 8 \\ 3 & 8 \le t \le 10 \end{cases}$$

$$5 + (4-5)u_2(t) + (2-4)u_6(t) + (3-2)u_8(t)$$

$$5 - u_2(t) - 2u_6(t) + u_8(t)$$

$\downarrow \mathcal{L}$

$$\boxed{\frac{5}{s} - \frac{e^{-2s}}{s} - 2\frac{e^{-6s}}{s} + \frac{e^{-8s}}{s}}$$

LRC - series

$L = 1$

$R = 7$

$C = \frac{1}{11}$