1 -

There will be one problem on solving a system of 1st order linear ODE's.
For example,

$$\frac{dx_1}{dt} = 1x_1(t) + 2x_2(t)$$

$$\frac{dx_2}{dt} = 4x_1(t) + 3x_2(t)$$

1) Find eigenvalues
2) Find eigenvectors
3) Find General Solution
4) Plot phase portrait

Solution.

1) Find eigenvalues:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

$$\Rightarrow (1-\lambda)(3-\lambda)-(8)$$

$$3-3\lambda-\lambda+\lambda^2-8$$

$$\lambda^2-4\lambda-5$$

$$(\lambda+1)(\lambda-5)$$

$$\lambda_1=-1 \qquad \lambda_2=5$$

2) Find eigenvectors:

$\lambda_1 = -1$

$$\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

$$\begin{bmatrix} 1-(-1) & 2 \\ 4 & 3-(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2\eta_1 + 2\eta_2 = 0$$

$$\eta_1 = -\eta_2$$

for $\eta_2 = 1$

$$\begin{bmatrix} -\eta_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 5$

$$\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

$$\begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4\eta_1 + 2\eta_2 = 0$$

$$2\eta_2 = 4\eta_1$$

$$\eta_2 = 2\eta_1$$

for $\eta_1 = 1$

$$\begin{bmatrix} \eta_1 \\ 2\eta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

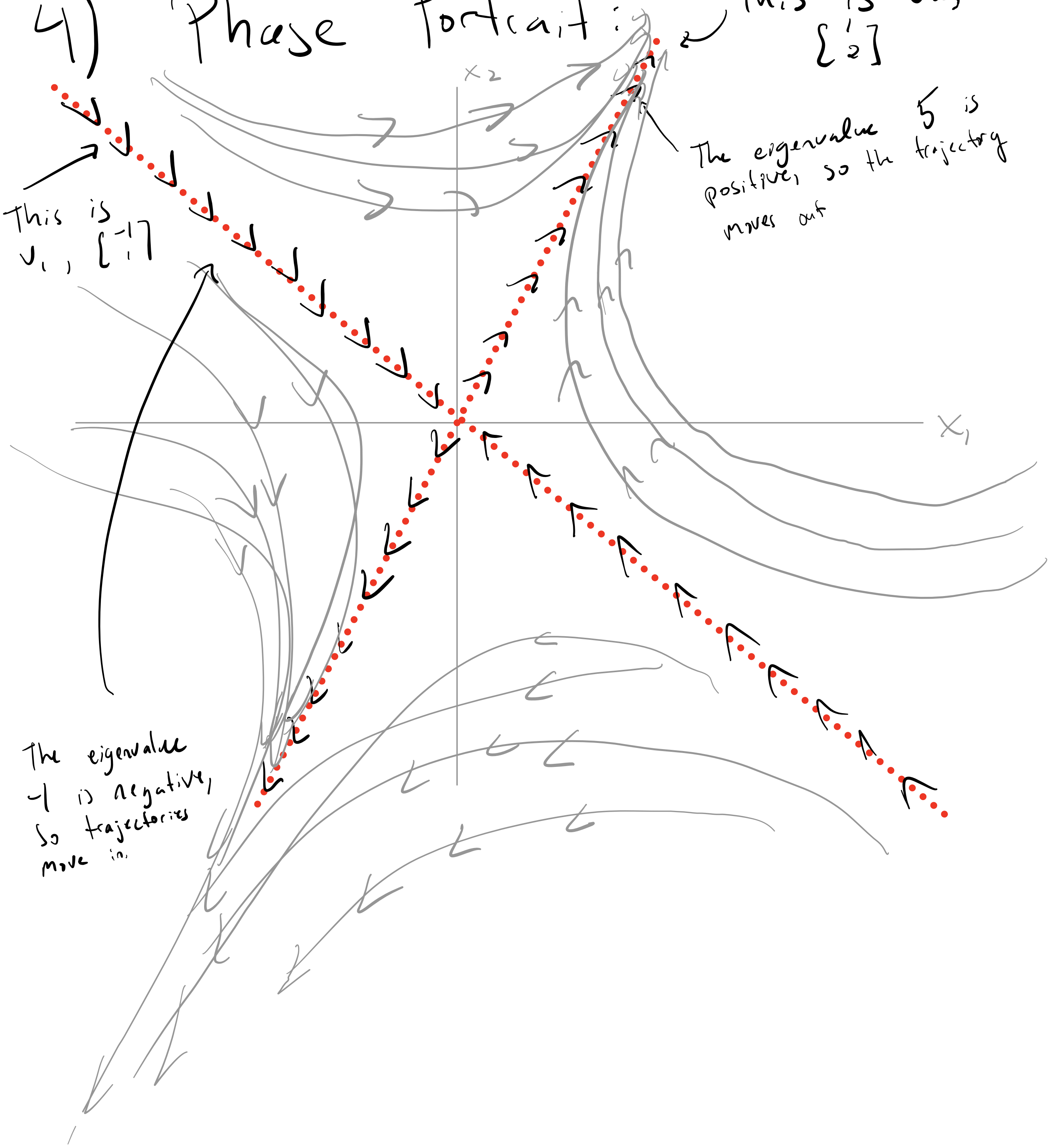
$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3) General Solution:

$$y(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$$

$$y(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4) Phase Portrait:



2 -
The second problem will ask you to solve the Initial Value Problem for a 3x3 matrix. You will know the eigenvalues and eigenvectors from the problem statement. The notation of the problem may seem unusual, but don't worry! Only a few computations are required. If you are doing a lot of math, you are probably over thinking this problem.

Q2 will look something like:

Q2. Solve the IVP $\frac{dx}{dt} = Ax(t)$, $x(0) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

where A is a constant matrix 3x3 satisfying:

i) $A \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0$

ii) $(A - 3I) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$,

iii) The vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue -1.

Again, we are actually told our eigenvalues and eigenvectors.

First, remember that to obtain eigenvectors, we have

$$(A - \lambda I)[v] = 0.$$

So, for

i) $A \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0 \Rightarrow (A - 0I) \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0$

eigenvalue $= 0$

eigenvector $= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

ii) $(A - 3I)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$

eigenvalue $= 3$

eigenvector $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

iii) It is given to us that

eigenvalue $= -1$

eigenvector $= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Here is how to solve:

General Solution:

$$x(t) = C_1 e^{0t}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_3 e^{-t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= C_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_3 e^{-t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

IVP.

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + C_2 e^{3(0)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_3 e^{-0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$0 = C_1 + C_2$$

$$0 = 2C_1 + C_3$$

$$2 = C_3$$

$$\underline{C_3 = 2}$$

$$0 = 2C_1 + 2$$

$$C_1 = -1$$

$$0 = C_1 + C_2$$

$$0 = -1 + C_2$$

$$C_2 = 1$$

So, the solution to the IVP is:

$$C_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\boxed{x(t) = -\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2e^{-t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

3 - This question has 3 parts:

$$y'' + \frac{1}{t-2} y' + 2y = e^t$$

1) What is the order?

2) Transform to a linear system of two first-order ODE's.

3) Find the maximum interval of existence of the solution of the ODE satisfying initial condition $y(0) = 1$, $y'(0) = 0$

Solution

1) $\boxed{\text{Order is } 2}$ (y'' is the highest order derivative of y)

2)

$$y'' + \frac{1}{t-2}y' + 2y = e^t$$

First, let

$u_1 = y$

$u_2 = y'$

Then,

$u_1' = y' = u_2$

$u_2' = y''$

$$u_2' = y'' = -\frac{1}{t-2}y' - 2y + e^t$$

$$u_2' = -\frac{1}{t-2}u_2 - 2u_1 + e^t$$

$$\boxed{\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{1}{t-2}u_2 - 2u_1 + e^t \end{aligned}}$$

3) Max interval:

$$y'' + \frac{1}{t-2} y' + 2y = e^t$$

a) Put in Standard Form
(This one already is)

b) Consider the function coefficients:

$\frac{1}{t-2} \rightarrow$ continuous on $(-\infty, 2)$ and $(2, \infty)$

$2 \rightarrow$ continuous on $(-\infty, \infty)$

$e^t \rightarrow$ continuous on $(-\infty, \infty)$

IVP is $y(0)=1$, $y'(0)=0$

The maximum overlapping interval containing 0 is

$$\boxed{(-\infty, 2)}$$