Math 2552 Quiz 5 Review

November 27 2023

1 Inverse Laplace with Convolution

Solve the initial value problem:

$$y'' + 9y = 2\sin 3t, y(0) = 1, y'(0) = 0$$

Solution

The laplace transform of the problem is

$$\left(s^2+9\right) Y - s = \frac{6}{s^2+9}.$$

We solve for $Y(s)$ and obtain

$$Y(s) = \frac{6}{\left(s^2+9\right)^2} + \frac{s}{s^2+9}.$$

The inverse laplace of the second term is clearly $\cos(3t)$. To take the inverse laplace of the first problem, we can use convolution. Convolution can be used when we have a function $H(s)$ which can be written as the product $F(s)G(s)$ of two functions. After taking the inverse laplace of the two functions, we can perform convolution $f(t) * g(t)$ to obtain $h(t)$.

The first term be written as the product of two laplace transforms:

$$\frac{6}{\left(s^2+9\right)^2} = \frac{2}{3}\frac{3}{(s^2+9)}\frac{3}{(s^2+9)}$$

We can then evaluate this using the convolution equation:

$$(f * g)(t) = \int_0^t f(u)g(t-u)du.$$

So,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{6}{\left(s^2+9\right)^2}\right] &= \frac{2}{3}(f * g)(t) \\
&= \frac{2}{3}\int_0^t \sin 3u \sin 3(t-u)du \\
&= \frac{1}{3}\int_0^t [\cos 3(2u-t) - \cos 3t]du \\
&= \frac{1}{3}\left[\frac{1}{6}\sin(6u-3t) - u\cos 3t\right]_0^t \\
&= \frac{1}{9}\sin 3t - \frac{1}{3}t\cos 3t. \qquad (1)
\end{aligned}$$

Our full solution is therefore

$$y(t) = -\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t + \cos 3t.$$

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2 Critical Points of Almost Linear Systems

Find the critical points of the following system:

$$x' = x(3 - x - 2y) \tag{1}$$
$$y' = y(4 - 2y - 4x). \tag{2}$$

Solution.

Note that there are two terms in (2): "y" and "$4 - 2y - 4x$". We can set both terms equal to 0 to find our solutions.

First, let us consider the first term in (2), for which we have $y = 0$. Let us substitute $y = 0$ into (1), and we solve:

$$x(3 - x - 2(0)) = 0$$
$$x(3 - x) = 0$$
$$x = 0, x = 3.$$

We have $x = 0$, $x = 3$, and $y = 0$. We can now construct combinations of these solutions to identify two fixed points: $(0, 0)$ and $(3, 0)$.

Second, we will consider the other term in (2): $4 - 2y - 4x$. Here, a solution for y is not a single real value but is instead a function of x. We may write

$$4 - 2y - 4x = 0$$
$$4 - 4x = 2y$$
$$2 - 2x = y.$$

We may now substitute y into (1) to solve for x, and we have

$$x(3 - x - 2y) = 0$$
$$x(3 - x - 2(2 - 2x)) = 0$$
$$x(3 - x - 4 + 4x) = 0$$
$$x(-1 + 3x) = 0$$
$$x = 0, x = \frac{1}{3}.$$

We may substitute each solution of x into $y = 2 - 2x$ to find the corresponding y-values:

$$y = 2,$$

and

$$y = \frac{4}{3}.$$

Therefore, we have the following two critical points: (0,2) and $(\frac{1}{3}, \frac{4}{3})$.
In total, we have identified four critical points:

$$(0, 0)$$
$$(3, 0)$$
$$(0, 2)$$
$$(\frac{1}{3}, \frac{4}{3}),$$

and we are done.

3 Approximate Linear Systems

Write the approximate linear system near (2,0) for the following nonlinear system:

$$x' = x(2 - 4x - 0.5y)$$
$$y' = y(1 - 3y - 0.75x)$$

Solution.

First, we can rewrite x' and y':

$$x' = 2x - 4x^2 - 0.5xy$$

$$y' = y - 3y^2 - 0.75xy$$

Then, we compute the Jacobian matrix J of this system:

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 2 - 8x - 0.5y & -0.5x \\ -0.75y & 1 - 6y - 0.75x \end{pmatrix}.$$

Now, we evaluate at (2,0):

$$\begin{pmatrix} 2 - 8(2) - 0.5(0) & -0.5(2) \\ -0.75(0) & 1 - 6(0) - 0.75(2) \end{pmatrix} = \begin{pmatrix} -14 & -1 \\ 0 & -0.5 \end{pmatrix}.$$

We may now write our solution:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -14 & -1 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} x - 2 \\ y \end{pmatrix}.$$

Our solution, in alternative notation you may have seen, can be written:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -14 & -1 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$

where we define $a = x - 2$ and $b = y$.

4 Stability of Linearized System

Determine the stability of the point (2,0)[1]

Solution.

Recall from the solution from the previous question is

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -14 & -1 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

We may now solve for the eigenvalues of our matrix. Remember that to solve for the eigenvalues for a matrix A, we first compute A - λI and then solve for the roots of a polynomial $ad - bc$.

For a 2 x 2 matrix, when either b or c is 0, our eigenvalues are equal to the diagonal elements (this is because $bc = 0$ so $ad - bc = ad - 0 = ad$.

[1] (2,0) in this case is not actually a fixed point to the system. You will generally write the approximating linear system near a fixed point, although this does not change anything with the solution process.

Therefore, our eigenvalues are -14 and -0.5. Since our eigenvalues are real, different, and negative, the system is **asymptotically stable**. If our eigenvalues were real, different, and positive, the system would be unstable. If both were real (but one is positive and one is negative), the system would be an unstable saddle point.